HILBERT'S AXIOMS OF PLANE ORDER

C. R. WYLIE, JR., Ohio State University

1. Introduction. Beyond the bare facts of the courses they will be called upon to teach, there are probably few things which can contribute more to the training of teachers of secondary school geometry than participation in a critical examination of the definitions and axioms of Euclid, and a comparison of them with a carefully developed modern equivalent, for instance the axiomatic system of David Hilbert.*

Of the various sets of axioms included in Hilbert's system, the axiom of parallels is in some ways the most interesting, opening up as it does the spectacular fields of non-euclidean geometry through its denial. However in another sense a careful scrutiny of the axioms of order affords a more profitable investment of time for those who have no lasting interest in any geometry but euclidean.

In the first place, the very existence of these axioms, devoted to such an obvious and familiar notion as *betweenness* serves to emphasize the fundamental modern attitude that geometry is an abstract science concerned with undefined elements whose properties are to be inferred from a set of consistent but otherwise essentially arbitrary assumptions, and not from any pseudo-isomorphism between the geometry elements and objects in the universe of experience. Secondly, it is in connection with the axioms of order that the logical structure of Euclid is weakest. In fact most of the fallacies of elementary geometry, such as the well-known "proofs" that all angles are right angles, and that all triangles are isosceles, spring immediately from the lack of clear-cut order relations.

The present note is essentially an expository account of the independence of these axioms, which may be of interest to those engaged in the training of secondary school teachers, and to those interested in the foundations of geometry, for some of the results obtained here are new, so far as the author is aware.

2. Hilbert's axioms. We shall consider exclusively the order axioms of Hilbert as they appear in the Carus Monograph,* The Foundations of Geometry. (*loc. cit.*) They are

1. If A, B, C, are points of a straight line, and B lies between A and C, then B lies between C and A.
2. If A and C are two distinct points of a straight line, then there exists at least one point B lying between A and C, and at least one point D, so situated that C lies between A and D.
3. Of any three points situated on a straight line, there is always one and only one which is between the other two.
4. Any four points A, B, C, D of a straight line can always be so arranged [*i.e.*, named] that B shall lie between A and C and also between A and D, and furthermore that C shall lie between A and D, and also between B and D.

* The Foundations of Geometry, David Hilbert; La Salle, Ill., 1938.

If we now define the segment AB to be the set of all points which are between A and B, we can add to the above axioms which define the notion of *betweenness* for points on a single line, the plane order axiom of Pasch

5. Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane of A, B, C and not passing through any of the points A, B, C. Then if the straight line a passes through a point of the segment AB it will also pass through either a point of the segment BC or a point of the segment AC.

3. Independence of the linear axioms. Considering the linear axioms 1–4 by themselves it is easy to establish their independence by concrete representations. For instance the following example shows that axiom 1 is independent of axioms 2–4.

Let the system consist of all the points on a line, and define "between A and B" to mean *actually between* if B is to the right of A, and *between but not the midpoint of* if B is to the left of A.

Similarly axioms 2, 3, 4 are shown to be independent by the following examples respectively. The system of points with coordinates $0, \pm 1, \pm 2, \cdots$ on a line with "between" defined to mean *actually between*. The system of all points on a line with "between" defined to mean *not between* [!]. The system of all points on a line with "between A and B" defined to mean *to the right of both A and B*.

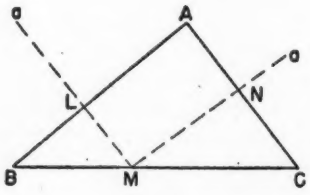


FIG. 1

4. Precise meaning of axiom 5. When axiom 5 is added to the four linear axioms certain dependencies are introduced. Before proceeding to a discussion of these it is desirable, however, to clarify a possible ambiguity in the meaning of this axiom. As stated, it is not entirely clear whether its alternatives are exclusive or not. That they are is an easy matter to demonstrate, and we now show that by appealing only to axiom 3 and the weak form of axiom 5, we may conclude that the line a which by hypothesis passes through a point of the segment AB can pass through a point in *only one* of the remaining segments BC or AC. In fact let the line a in Fig. 1, not passing through any of the points A, B, C, pass through points L, M, N of the segments AB, BC, and AC respectively. By axiom 3 one of the points L, M, N, say M, lies between the other two.* Now consider the three non-collinear points L, N, A. The straight line BC passes

* Henceforth we shall indicate such a "betweenness" relationship by writing simply (LMN).

through the point M of the segment LN. Therefore it must pass through a point in *at least* one of the segments NA or LA. But this is impossible, for by axiom 3 since N lies between A and C, C cannot simultaneously be between N and A, and for the same reason B cannot lie between L and A. This contradiction of the weak form of axiom 5 establishes the result.

5. Dependence of axiom 1. We now prove that axiom 1 is a consequence of axioms 2, 3, and 5. Let (AXB) be true on a line l, Fig. 2. Choose C not on l and Y on the line AC so that (ACY), (axiom 2). Now consider the line XY in relation to the three non-collinear points A, B, C. It passes through the point X, of the segment AB. It must therefore pass through a point of the segment BC

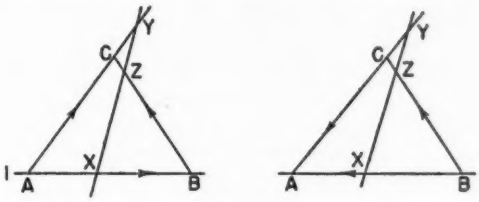


FIG. 2

or a point of the segment AC, by axiom 5. By axiom 3 it cannot pass through a point of AC, therefore it must pass through a point, say Z, of the segment BC. Now consider this same line in relation to the three points B, C, A. It surely passes through a point, namely Z, of the segment BC. It cannot pass through a point of CA, hence it must pass through a point of the segment BA. This is possible if and only if X is between B and A.

6. Dependence of axiom 4. Axiom 4 is also a consequence of axioms 2, 3, and 5, as was first pointed out by E. H. Moore* in 1902. His proof though entirely straightforward was somewhat lengthy, and a shorter proof may be of interest here. This seems doubly appropriate since in subsequent editions of Hilbert's *Foundations* this result is still not noted.†

We base our proof upon the following series of observations.

LEMMA: *Of four points, A, B, C, D, situated on a line l, no one can be "between" only once.*

We assume the contrary and suppose that of the four points A, B, C, D on l, Fig. 3, the point B lies between A and C and between no other pair. Choose E not on l, and on CE choose F so that (CEF), axiom 2. Now consider the line BF in relation to the three points A, C, E. It passes through the point B of the

* E. H. Moore, The projective axioms of geometry, Trans. Amer. Math. Soc., 1902.

† It is, of course, to be found in various places outside of Moore's paper. For instance J. W. Young makes explicit mention of this dependence in his book, Fundamental Concepts of Algebra and Geometry.

segment AC. Since by axiom 3 it cannot pass through a point of the segment CE, it must pass through a point, say G, of AE. Now consider this same line in relation to the three points A, E, D. It passes through the point G of the segment AE. It cannot pass through a point of the segment AD, since by hypothesis B is only between A and C on l. Therefore it must pass through a point of ED, say H. Now finally consider this same line in relation to the points E, D, C. It passes through a point of ED as we have just seen. However it cannot pass through a point of DC by hypothesis, and by axiom 3 it cannot pass through a point of EC. This contradicts axiom 5 and establishes our result.

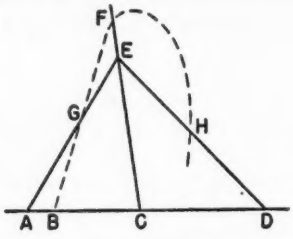


FIG. 3

As a first corollary of this lemma we conclude that of our points on a line, no one can be "between" three times. For there are only four combinations of three points each, existing in the set of points A, B, C, D, and by axiom 3 only one point in each triad can be "between." Hence there are only four possible "betweeness" relations insofar as the central term is concerned. If then, one point is the middle point in three of these, some other point must be "between" only once, which contradicts the lemma.

As a second corollary we observe that of four points on a line, two are necessarily "between" exactly twice, and two are "between" not at all.

The proof of axiom 4 as a consequence of axioms 2, 3, 5, and 1 (now known to be a theorem independent of axiom 4) follows at once from the second corollary. Of the four points A, B, C, D on a line l let the points which are never "between" be called A and D, the others B and C. Then of the respective sets of relations

$$(ABC) \quad (ABD) \quad (CBD)$$

and

$$(ACB) \quad (ACD) \quad (BCD)$$

two and only two of each set must be true. If first, (ABC) is true, then (ACB) is necessarily false by axiom 3. Hence (ACD) and (BCD) are true, and since the last is true, (CBD) is false, leaving (ABD) true. The points are then named according to axiom 4. If on the contrary (ABC) is false, then inevitably (ABD), (CBD), (ACB), and (ACD) are true, and if the names B and C are interchanged, the ordering of the axiom is again accomplished.

7. Independence of axioms 2, 3, and 5. The three axioms that remain may now be shown to be independent by the following concrete representations.

For the system consisting of three non-collinear points and the three lines through them, with "between" defined to mean *actually between*, axiom 2 is false and axioms 3 and 5 are (vacuously) true.

In the ordinary euclidean plane with "between" defined to mean *collinear with* axioms 2 and 5 are true but axiom 3 is false.

In the ordinary euclidean plane with "between" defined to mean *above both* on all vertical lines, and *to the right of both* on all other lines, axioms 2 and 3 are valid but axiom 5 is false.

8. A new form of axiom 5. We have now completed our exposition of the dependence and independence of the axioms of betweenness in the specific form in which Hilbert presented them. Numerous other sets of axioms have been devised to define this same notion. In general these differ substantially from Hilbert's system, and we do not propose to consider them here. There is however one other system almost identical with that of Hilbert which we shall touch on in conclusion.

Suppose we keep the axioms as quoted in paragraph I intact except that in axiom 5 we change the last phrase to read " . . . it will also pass through either a point of the segment BC *or a point of the segment* CA." The nature of this new set of axioms must be regarded as a new problem, entirely independent of those observations which led us to our conclusions concerning the dependence of Hilbert's own set. It may well be that axioms dependent in the first formulation will now be independent or vice-versa. And the proofs of such relations as are true in both systems may differ widely in their character.

9. Dependence of the new axioms. We shall not explore this new system in detail. The final results are curiously enough the same here as before: axioms 1 and 4 are consequences of axioms 2, 3, and 5. The proof of this follows roughly the outline used above. For preliminary clarification we can prove a strong form of axiom 5 precisely as we did in treating Hilbert's version of the axioms.

When we attempt to establish the dependence of axiom 1, however, we must devise an entirely new form of proof. In fact it must be based upon the lemma of paragraph 6 and its corollaries, instead of being independent of them. We omit the new proof of this lemma which follows *mutatis mutandis* from the earlier proof, and give only the new proof of the dependence of axiom 1.

Suppose that on a line l, (AXC) is true while (CXA) is false. Choose B not on l, Fig. 4, and choose Y so that (BYA), axiom 2. Now consider the line XY in relation to the points B, A, C. It passes through the point Y of the segment BA, and the point X of the segment AC. Therefore by the strong form of axiom 5* it cannot pass through a point of CB. Now choose D so that (AYD) and consider the same line in relation to the points A, D, C. It passes through a point

* Note that in all our discussion of Hilbert's axioms we did not need to use the strong form of axiom 5, and only called attention to it in order to secure a clearer understanding of the system.

of AD, namely Y, does not pass through a point of CA, since by hypothesis (CXA) is false, hence must pass through a point, say Z, of DC. Finally consider this line in relation to the points D, C, B. It passes through the point Z of DC, does not pass through a point of CB, as pointed out above, and therefore must pass through a point of BD. This is possible if and only if Y is between B and D. But then in the range A, Y, B, D, the point Y is between three times which is impossible by the first corollary. This contradiction establishes the result.

The proof that axiom 4 is dependent is literally identical in each system and we pass over it without mention.

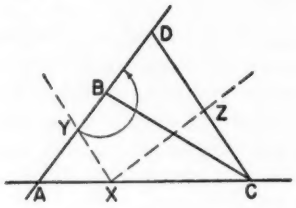


FIG. 4

10. Conclusion. We have considered as a separate problem the modified system that we have just discussed, not only as a simple experiment designed to make clear how a very slight change in an axiom may completely alter the character of a proof, but also to illumine a careless treatment of these axioms that is sometimes encountered. Axiom 5 which introduced the dependencies we have been discussing is sometimes loosely stated "A straight line which meets one side of a triangle and lies in the plane of the triangle meets at least one other side of the triangle."

Until a triangle is defined, this statement is meaningless. If the word triangle is taken to mean the configuration shown in Fig. 5a, associated as it were with vector addition, the axiom is simply an ill-stated version of Hilbert's axiom 5.

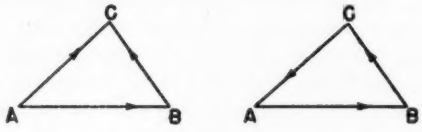


FIG. 5

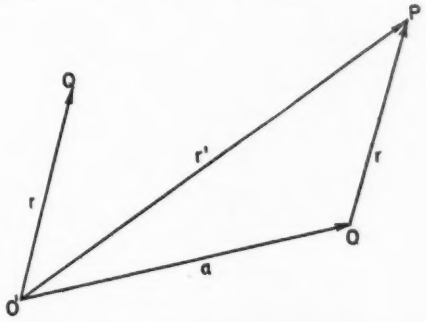
If, equally naturally, the word triangle is taken to imply the cyclic configuration shown in Fig. 5b we have the second version of axiom 5. If finally the word triangle is defined in the sense of elementary high school geometry, we pemit either of the above definitions, and commit the paralogism of incorporating axiom 1 in the definition of a term employed in another axiom. Small wonder that we can then prove it dependent!

MOTION WITH RESPECT TO MOVING AXES

J. W. CAMPBELL, University of Alberta

A paper by the author under a similar title has appeared previously.* The development there was in coordinate notation, and in the present paper the same results are obtained using vector notation for the discussion. The vector treatment also throws some additional sidelights on the problem.

1. Newtonian functions. For the sake of clarity we make a few explanations. In Newtonian mechanics when rates of change of quantities are referred to they are to be taken with respect to fixed or inertial axes. This is implied by the fact that Newton's laws of motion apply fundamentally only to such axes. It is often convenient, however, to use moving axes, and this necessitates extra care. Accordingly we introduce the idea of a Newtonian function, by which we shall mean a function which involves rates of change only with respect to fixed or inertial axes. Since for vector notation, however, we need only reference points and not axes, we can replace *axes* by *points* in the above definition when vector notation is to be used. But when interpreting results in component form we must take components with respect to fixed axes.



To illustrate what we mean, let O' be a fixed point and O a moving point whose position with respect to O' is $\mathbf{a}$; let P be a point whose position with respect to O is $\mathbf{r}$, and with respect to O' is $\mathbf{r}'$, so that

$$\mathbf{r}' = \mathbf{r} + \mathbf{a}; \tag{1}$$

also, for what follows, let O' be a former position of O.

Now let ϕ be a vector function of position with respect to any chosen reference point. Then $\phi(\mathbf{r}')$ is the function associated with P and with respect to O'; $\phi(\mathbf{r})$ is the function associated with P and with respect to O, or the function associated with Q and with respect to O'.

* J. W. Campbell, Motion re moving axes, Trans. Roy. Soc. of Can., Ser. III, vol. XXXVI, pp. 1–6.

But $\phi(\mathbf{r}')$ may depend on $\dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots$ as well as on $\mathbf{r}'$ and t, and so we write

$$\phi(\mathbf{r}') = \phi(\mathbf{r}', \dot{\mathbf{r}}', \ddot{\mathbf{r}}' \cdots ; t).$$

In that case, if ϕ is a Newtonian function,

$$\phi(\mathbf{r}) = \phi(\mathbf{r}, \dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots ; t).$$

For any rates of change occurring must be with respect to the fixed point with which the moving point instantaneously coincides, and the rates of change with respect to all fixed points are equal.

As an illustration consider the particular case of angular momentum. If a mass m were located at P its angular momentum about O' would be $\dot{\mathbf{r}}' \times (m\dot{\mathbf{r}}')$ and its angular momentum about O would be $\mathbf{r} \times (m\dot{\mathbf{r}}')$, not $\mathbf{r} \times (m\dot{\mathbf{r}})$. The latter would not be a Newtonian function.

Inasmuch, then, as the other arguments do not change with change of reference point, we shall for simplicity retain the position argument only and so we write $\phi(\mathbf{r}')$ and $\phi(\mathbf{r})$.

2. Derivatives. It follows from (1) that

$$\phi(\mathbf{r}') = \phi(\mathbf{r} + \mathbf{a}). \tag{2}$$

Now the Taylor's expansion

$$\begin{aligned} \phi(x + h, y + k, z + l) &= \phi(x, y, z) \\ &+ \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right) \phi(x, y, z) \\ &+ \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right)^2 \phi(x, y, z) \\ &+ \cdots \end{aligned} \tag{3}$$

can be written in vector notation as

$$\phi(\mathbf{r} + \mathbf{a}) = \phi(\mathbf{r}) + (\mathbf{a} \cdot \nabla)\phi(\mathbf{r}) + \frac{1}{2!} (\mathbf{a} \cdot \nabla)^2 \phi(\mathbf{r}) + \cdots .^* \tag{4}$$

Hence equation (2) can be written

$$\phi(\mathbf{r}') = \phi(\mathbf{r}) + (\mathbf{a} \cdot \nabla)\phi(\mathbf{r}) + \frac{1}{2!} (\mathbf{a} \cdot \nabla)^2 \phi(\mathbf{r}) + \cdots . \tag{5}$$

Differentiating with respect to t we have

$$\begin{aligned} \frac{d}{dt} \phi(\mathbf{r}') &= \frac{d}{dt} \phi(\mathbf{r}) + (\dot{\mathbf{a}} \cdot \nabla + \mathbf{a} \cdot \dot{\nabla})\phi(\mathbf{r}) \\ &+ \text{terms involving } \mathbf{a} \text{ as a "factor."} \end{aligned}$$

* See J. G. Coffin, Vector Analysis, p. 131.

Proceeding to the limit as $\mathbf{a}\to 0$, we have

$$\frac{d}{dt}\phi(\mathbf{r}') = \frac{d}{dt}\phi(\mathbf{r}) + (\dot{\mathbf{a}}\cdot\nabla)\phi(\mathbf{r}). \tag{6}$$

Now since $\phi(\mathbf{r}')$ is the function associated with P and with respect to O', the left-hand side of (6) is the rate of change of ϕ with respect to the fixed point with which O instantaneously coincides. It is therefore the derivative in the Newtonian sense and so may be called the Newtonian derivative.

On designating the Newtonian derivative by a subscript N we have

$$\left(\frac{d}{dt}\right)_N \phi(\mathbf{r}) = \frac{d}{dt}\phi(\mathbf{r}) + (\dot{\mathbf{a}}\cdot\nabla)\phi(\mathbf{r}).$$

In more compact notation, if $\phi(\mathbf{r})$ is denoted by $\mathbf{V}$ and if $\mathbf{u}$ is the velocity of O with respect to a fixed point, then

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}} + (\mathbf{u}\cdot\nabla)\mathbf{V}. \tag{7}$$

To see the meaning of $\dot{\mathbf{V}}$, recall that $\phi(\mathbf{r})$ can be interpreted as the function associated with Q and with respect to O'. Therefore $\dot{\mathbf{V}}$ is the rate of change of $\mathbf{V}$ *as if O were fixed*, and the expression for $\dot{\mathbf{V}}$ is known from classical theory. Thus, if $\mathbf{V}=(V_1, V_2, V_3)$ and the axes are turning with the angular velocity $\boldsymbol{\omega}$, then

$$\dot{\mathbf{V}} = \dot{\mathbf{V}}_f + \boldsymbol{\omega}\times\mathbf{V}, \tag{8}$$

where

$$\dot{\mathbf{V}}_f = \dot{V}_1\mathbf{i} + \dot{V}_2\mathbf{j} + \dot{V}_3\mathbf{k}. \tag{9}$$

Therefore

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}}_f + \boldsymbol{\omega}\times\mathbf{V} + (\mathbf{u}\cdot\nabla)\mathbf{V}. \tag{10}$$

The first term on the right hand side of (10) gives the rate of change of $\mathbf{V}$ as if the axes were fixed, the second term gives the rate of change due to turning of axes, and the third term the rate of change due to translation.

Written in component form this equation gives

$$\begin{aligned}
\dot{V}_{1N} &= \dot{V}_1 - \omega_3 V_2 + \omega_2 V_3 + u_1\frac{\partial V_1}{\partial x} + u_2\frac{\partial V_1}{\partial y} + u_3\frac{\partial V_1}{\partial z},\\
\dot{V}_{2N} &= \dot{V}_2 - \omega_1 V_3 + \omega_3 V_1 + u_1\frac{\partial V_2}{\partial x} + u_2\frac{\partial V_2}{\partial y} + u_3\frac{\partial V_2}{\partial z},\\
\dot{V}_{3N} &= \dot{V}_3 - \omega_2 V_1 + \omega_1 V_2 + u_1\frac{\partial V_3}{\partial x} + u_2\frac{\partial V_3}{\partial y} + u_3\frac{\partial V_3}{\partial z}.
\end{aligned} \tag{11}$$

These results agree with those of the previous paper.

3. Remarks. It might be thought that the right hand side of (7) would introduce terms in $\dot{\mathbf{r}}$, but such is not the case. We can conclude this from the fact that the Newtonian derivative is a Newtonian function, but we can also see it analytically.

For (7) can be written in the form

$$\dot{\mathbf{V}}_N = \frac{D'V_1}{D't}\mathbf{i} + \frac{D'V_2}{D't}\mathbf{j} + \frac{D'V_3}{D't}\mathbf{k} + \omega \times \mathbf{V} \tag{12}$$

when $D'/D't$ is an operation such that

$$\begin{aligned}\frac{D'V_1}{D't} &= \frac{dV_1}{dt} + u_1\frac{\partial V_1}{\partial x} + u_2\frac{\partial V_1}{\partial y} + u_3\frac{\partial V_1}{\partial z} \\ &= (u_1 + \dot{x})\frac{\partial V_1}{\partial x} + (u_2 + \dot{y})\frac{\partial V_1}{\partial y} + (u_3 + \dot{z})\frac{\partial V_1}{\partial z} \\ &\quad + \text{terms not involving } x, y, z.\end{aligned} \tag{13}$$

That is

$$\frac{D'V_1}{D't}\mathbf{i} + \frac{D'V_2}{D't}\mathbf{j} + \frac{D'V_3}{D't}\mathbf{k} = (\dot{\mathbf{r}}'\cdot\nabla)\mathbf{V} + \text{terms in } t, \mathbf{r}, \dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots. \tag{14}$$

Therefore by (12) and (13) $\dot{\mathbf{V}}_N$ contains no terms in $\dot{\mathbf{r}}$.

In particular, if $\mathbf{V} = \mathbf{r}$ we obtain from (10) and (12)

$$\dot{\mathbf{V}}_N = \dot{\mathbf{r}}_f' + \omega \times \mathbf{r} = \dot{\mathbf{r}}'.$$

It is worth noting that it is only in the cases where $\mathbf{V}$ does not depend explicitly on the derivatives $\dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots$ that (13) can be written in the form

$$\frac{D'V_1}{D't} = \frac{\partial V_1}{\partial t} + (u_1 + \dot{x})\frac{\partial V_1}{\partial x} + (u_2 + \dot{y})\frac{\partial V_1}{\partial y} + (u_3 + \dot{z})\frac{\partial V_1}{\partial z}.$$

This is analogous to, but not identical with, the hydrodynamical equation

$$\frac{DV_1}{Dt} = \frac{\partial V_1}{\partial t} + u_1\frac{\partial V_1}{\partial x} + u_2\frac{\partial V_1}{\partial y} + u_3\frac{\partial V_1}{\partial z}.$$

It therefore follows that the general dynamical equations for axes moving in any way are entirely independent of the equations of hydrodynamics in spite of some apparent analogy of form.

4. Functions of several points. In the foregoing discussion ϕ has been regarded as a function of a single point $\mathbf{r}$. It might, however, depend on a set of points r_i $(i = 1, \cdots, n)$. In that case there will be a variation of ϕ associated with each one of these points when the origin moves. By a sequential application of the Taylor's series process it will be found that if

$$\mathbf{V} = \phi(\mathbf{r}_1, \cdots, \mathbf{r}_n),$$

then

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}}_f + \boldsymbol{\omega} \times \mathbf{V} + \left(\mathbf{u} \cdot \sum_{i=1}^{n} \nabla_i\right)\mathbf{V}, \tag{15}$$

where ∇_i is the operator associated with $\mathbf{r}_i$.

A special case of importance may arise. In the expression $\phi(\mathbf{r}_1, \cdots, \mathbf{r}_n)$ the $\mathbf{r}$'s may occur in pairs and be subtracted. In that case if we put $\mathbf{r}_i' = \mathbf{r}_i + \mathbf{a}$, the $\mathbf{a}$'s will all cancel in ϕ and so we shall have

$$\phi(\mathbf{r}_1', \cdots, \mathbf{r}_n') = \phi(\mathbf{r}_1, \cdots, \mathbf{r}_n).$$

On differentiation we find that

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}}_f + \boldsymbol{\omega} \times \mathbf{V}, \tag{16}$$

and so the term $(\mathbf{u} \cdot \Sigma\nabla_i)\mathbf{V}$ is zero. This might also have been inferred from the fact that when two $\mathbf{r}$'s occur as a pair in ϕ and are subtracted, the ∇'s corresponding to these $\mathbf{r}$'s will cancel.

Vector functions of position are, therefore, divided into two classes, *viz.*, those that are affected by a translation of the origin and those that are not. When they are not affected by a translation of the origin the Newtonian derivative is given by (16).

If a function is explicitly a function of $\mathbf{r}$'s with the $\mathbf{r}$'s subtracted in pairs then we can conclude that a translation of the origin has no effect on the function and so we can use (16). But sometimes only one $\mathbf{r}$ appears explicitly and yet it is known that a translation of the origin has no effect on the function. In such cases, also, the Newtonian derivative is given by (16); another $\mathbf{r}$ is implicitly present to make this the correct result.

An example is the velocity vector. The Newtonian derivative of $\mathbf{r}$ is

$$\mathbf{v} = \dot{\mathbf{r}}_N = \dot{\mathbf{r}}_f + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{u}\dot{\phi},$$

and this is explicitly a function of $\mathbf{r}$. But the velocity is obviously independent of the position of the origin and so we can write

$$\dot{\mathbf{v}}_N = \dot{\mathbf{v}}_f + \boldsymbol{\omega} \times \mathbf{v}.$$

In such cases it is not necessary to determine the $\mathbf{r}$ which appears implicitly, but in the case of the velocity vector it is not hard to find. For if $\mathbf{r}_1$ is a point on the instantaneous axis of the frame then $\mathbf{u} = -(\dot{\mathbf{r}}_1)_f - \boldsymbol{\omega} \times \mathbf{r}_1$, and therefore

$$\mathbf{v} = (\dot{\mathbf{r}} - \dot{\mathbf{r}}_1)_f + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_1).$$

THE GEOMETRIC METHOD IN MATHEMATICAL STATISTICS

D. D. KOSAMBI, Fergusson College, Poona, India

1. Introduction. R. A. Fisher was the first to make use of n-dimensional geometry in the derivation of certain distributions. A direct approach is always possible, and is even to be preferred, by the use of the Fourier transform, with or without the transformation theory of positive definite quadratic forms. Nevertheless, the geometric method offers great advantages in brevity, clarity, and insight. It is in no way inferior in rigor to any other method and finally, its applications extend to a greater number of the distributions used in small-sample theory than is realized.

2. Geometric preliminaries. In an euclidean space of $n \geqq 1$ dimensions with coordinates $(x_1, x_2, \cdots, x_n)$ for a generic point x, the distance d between two points x and x' is given by

$$d^2 = \sum (x_i - x_i')^2. \tag{1}$$

In all summations, unless otherwise specified, the range is over values 1 to n of the index. This definition of distance, combined with the usual methods in use for 3-dimensional euclidean geometry, suffices to derive most of the formulas we need.

The vector $x'-x$ has direction components $a_i=x_i'-x_i$, $i=1, 2, \cdots, n$, so that the square of its length is $d^2=\sum a_i^2$, and the direction cosines $\alpha_i=a_i/d$, with $\sum \alpha_i^2=1$. The angle θ between two directions is given by $\cos\theta=\sum \alpha_i\alpha_i' = \sum a_i a_i'/dd'$.

A hyperplane in n-space is represented by a linear equation

$$\sum a_i x_i = y. \tag{2}$$

The coefficients a_i are the direction components of the normal to the plane. The perpendicular distance from a point x to the plane is $p=(\sum a_i x_i - y)/\sqrt{\sum a_i^2}$, where the proper sign is to be taken in the square root so as to make this distance positive. When the direction cosines α_i are used in place of a_i in equation (2), y is itself the perpendicular distance from the origin to the plane. The foot of the perpendicular to the plane from the point x is $\bar{x}$, given by

$$\bar{x}_i = x_i - \lambda a_i, \quad \text{where} \quad \lambda = (\sum a_i x_i - y)/\sum a_i^2. \tag{3}$$

The hypersphere of radius r centered at the origin is

$$\sum x_i^2 = r^2 \quad \text{or} \quad \sum x_i^2 \leqq r^2, \tag{4}$$

of which the first represents the surface, and the second the volume of the sphere. A plane section of the n-sphere is always an $n-1$ sphere where we understand by the one-dimensional sphere, the line-interval $(-r, r)$, with "surface" $x=\pm r$, the two-dimensional sphere being the circle, and so on. The volume of

the n-sphere is $2\pi^{n/2}r^n/n\Gamma(n/2)$. It suffices for our purpose that this volume should be cr^n, which is equivalent to the statement that the volume element for purposes of integration is

$$dV_n = dx_1 dx_2 \cdots dx_n = r^{n-1} f(\theta_1\theta_2 \cdots \theta_{n-1}) dr d\theta_1 d\theta_2 \cdots d\theta_{n-1}. \tag{5}$$

The actual form of f for any given n may be derived by simple extension of the 3-dimensional spherical polar coordinate formula.

3. The normal distribution and euclidean n-space. The most important basic distribution [important because of the "central limit theorem" which shows that it is the limiting form of the distribution of an average from any fairly general type of population] is the normal distribution, where the elementary probability is given by $(1/\sigma\sqrt{2\pi}) \exp -(x-\mu)^2/2\sigma^2 dx$. By change of origin and scale, we can always take the population mean μ as zero, and the population variance σ^2 as unity; this we shall call standard measure.

For several normally distributed standard independent variables $x_1, \cdots, x_n$, the elementary probabilities are compounded by multiplication to give

$$\begin{aligned} dP &= \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} dx_1 \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} dx_2 \cdots \frac{1}{\sqrt{2\pi}} e^{-x_n^2/2} dx_n, \\ &= \frac{1}{(\sqrt{2\pi})^n} e^{-r_n^2/2} dV_n. \end{aligned} \tag{6}$$

The basis of the geometrical method as applied to mathematical statistics is that this probability density depends only upon r_n^2, *i.e.*, it is isotropic in the n-space. Moreover, this is a characteristic property of the normal distribution. For, if the elementary probability were $f(x)dx$, and if we should ask ourselves when a relation of type

$$f(x_1)f(x_2) \cdots f(x_n) dx_1 dx_2 \cdots dx_n = \phi(r_n)\, dV_n, \tag{7}$$

would be valid, we should obtain the functional equation

$$f(x_1)f(x_2) \cdots f(x_n) = \phi(\sqrt{\textstyle\sum} x_i^2), \tag{8}$$

to hold identically in x. Setting $x_2 = \cdots = x_n = 0$, we get $\phi(x_1) = f(0)^{n-1} f(x_1)$. The functional equation is thus equivalent to $f(x)f(y) = cf(\sqrt{x^2+y^2})$, which is known to have no continuous solutions except $f(x) = ae^{bx^2}$. This sort of argument is followed, for example, in the deduction of Maxwell's law in the kinetic theory of gases. If, now, the total range for each x_i is $-\infty, +\infty$, we have $b<0$, say $b = -1/2\sigma^2$, to make the total probability unity, $a = 1/\sigma\sqrt{2\pi}$. If the range be restricted, other distributions, including the uniform distribution, are possible; a fact that is forgotten by all who use this derivation for theoretical physics.

Not only may we build up the normal distribution in n dimensions from n independent individual distributions, but the process may also be reversed so

that one or more dimensions can be cut off as necessary. *Orthogonality is the geometric equivalent of independence in statistics.* We utilize this after performing linear transformations to new sets of orthogonal axes; for our purpose, dealing only with normal distributions in standard measure, rotations alone suffice. The effective number of dimensions in a given problem are called degrees of freedom by R. A. Fisher, who uses the letter n to denote this number. We shall use n for the original number of dimensions, keeping in mind that a random sample of n from an infinite basic population represents such an n-dimensional distribution as in (6). The degrees of freedom not in use are to be integrated out. Contrary to dynamical usage, the effective number of degrees of freedom in a statistical problem represents that number of dimensions in which the coordinates are free under the statistical conditions imposed only in the limited sense that a definite probability necessarily attaches to each region of the sub-space.

4. The distribution of the mean and variance in samples from normal. The χ^2-distribution. Let $y=\sum a_i x_i$ be a linear combination of n standard normal variables $x_1, \cdots, x_n$. To derive the distribution of y, we rotate so that one axis lies along the normal to the family of hyperplanes $y=\sum a_i x_i$, or with respect to the new axes, $y=$const., and the remaining $n-1$ lie in the plane $\sum a_i x_i = 0$. Taking $r=y/\sqrt{\sum a_i^2}$ we may split up the distance from the origin to a point x as $r_n^2=r_1^2+r_{n-1}^2$. That is, r_1 is the distance from the origin to the foot of the perpendicular to the particular hyperplane of the family $\sum a_i x_i = y$ which passes through the point $x=(x_1, \cdots, x_n)$; r_{n-1} is the distance in the plane from the foot of the perpendicular to the point x. The elementary probability of (6) may, therefore, be expressed as

$$(9) \qquad dP = \frac{1}{(\sqrt{2\pi})^n} e^{-r_n^2/2} dV_n = \frac{1}{\sqrt{2\pi}} e^{-r_1^2/2} dr_1 \cdot \frac{1}{(\sqrt{2\pi})^{n-1}} e^{-r_{n-1}^2/2} dV_{n-1}.$$

As the r_1 component alone interests us here, we may eliminate the rest by integration over the whole of V_{n-1}. Therefore, r_1 is normally distributed in standard measure. Hence, $y=r_1\sqrt{\sum a_i^2}$ is normal with zero mean and variance $\sum a_i^2$. Replacing $x_i+\mu_i$ by x_i, to pass from standard measure to the general normal distribution, we obtain:

THEOREM 1. *If n independent variates $x_1, \cdots, x_n$ are normally distributed with means μ_i and variances σ_i^2, any linear combination thereof $\sum a_i x_i$ is also normally distributed with mean $\sum a_i \mu_i$ and variance $\sum a_i \sigma_i^2$.*

On the other hand, we might have integrated out r_1 to concentrate upon r_{n-1}, noting that the resulting distribution in V_{n-1} would be normal, the original n-variables being now restricted to lie in a hyperplane, *i.e.*, to obey one identical linear restriction of type $\sum a_i x_i =$const. As r_{n-1} is to be measured from the foot of the perpendicular from the origin of n-space, formula (9) may be applied to give

THEOREM 2. *If upon n originally independent standard normal variables a linear restriction $\sum a_i x_i = b$ is imposed, we obtain a normal distribution in $n-1$ variables: $dP = c \exp -r_{n-1}^2/2 \cdot dV_{n-1}$, where*

$$r_{n-1}^2 = \sum (x_i - \bar{x}_i)^2, \qquad \bar{x}_i = ba_i / \sum a_i^2. \tag{10}$$

If we abandon standard measure, putting $x_i - \mu_i$ for x_i, we should have to replace b by $b + \sum a_i \mu_i$. Thus, $\bar{x}_i$ shifts correspondingly except in the special case where all these separate additions cancel out in each bracket $(x_i - \bar{x}_i)$. To this end it is necessary and sufficient that

$$\mu_i = \frac{a_i \sum a_i}{\sum a_i^2}. \tag{11}$$

Since in sampling problems $\mu_i = \mu$, $i = 1, 2, \cdots, n$, we have

THEOREM 3. *For random sampling from a normal population, the sole linear restriction independent of the population mean is of type $(x_1 + \cdots + x_n)/n = constant$.*

That is, in general, the population mean being unknown, we may measure from the sample mean $m = \sum x_i/n$ with the loss of a single degree of freedom. Moreover there is no other linear sample function from which such measurement may be made independent of the population mean. The statistic m has the further advantage that it is normally distributed, according to Theorem 1, with expectation equal to the population mean, and variance σ^2/n. Thus it is *unbiassed*, and using results and terminology due to Fisher, since no other estimate of μ can have a smaller variance than σ^2/n, m is the most *efficient* such estimate; finally, as $n \to \infty$, the probability for $m \neq \mu$ tends to zero, so that the estimate is consistent.

For the χ^2 and other tests, we need the distribution of the sum of squares of n normal standard variables, our r_n^2. That is,

$$P(r_n^2 \leqq t) = \int_{\sum x_i^2 \leqq t} \cdots \int e^{-(x_1^2 + \cdots + x_n^2)/2} dx_1 \cdots dx_n. \tag{12}$$

This is evaluated at once in spherical polar coordinates, integrating over thin spherical shells centered at the origin. This gives

$$P(r_n^2 \leqq t) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)(\sqrt{2\pi})^n} \int_0^{\sqrt{t}} e^{-r^2/2} r^{n-1} dr. \tag{13}$$

Transforming by $u = r^2$, $r = \sqrt{u}$, $dr = du/2\sqrt{u}$, we get

$$P(r_n^2 \leqq t) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_0^t e^{-u/2} u^{n/2-1} du. \tag{14}$$

Provided the population mean is zero, this χ^2 or incomplete gamma function distribution holds even for a change of scale, in particular for $\sum x_i^2/\sigma^2$. The expection of $u=r_n^2$ is easily calculated by multiplying under the sign of integration by u and integrating to infinity. This amounts to replacing the exponent n by $n+2$, so that the result must be $2\cdot 2^{n/2}\Gamma(n/2+1)/2^{n/2}\Gamma(n/2)=n$. Therefore, r_n^2/n furnishes, when the population mean is known, an unbiassed estimate of σ^2. For unknown population mean, we apply the findings of Theorem 3 to obtain

THEOREM 4. *For random sampling from the same normal population,* $\sum(x-m)^2$ *with* $m=\sum x_i/n$ *has the* χ^2 *distribution with* $n-1$ *degrees of freedom, its expectation being* $(n-1)$.

To estimate the population variance without bias, we must divide $\sum(x-m)^2$ by $n-1$, and not by n.

5. The Student-Fisher *t*-distribution. The next step is to derive statistics independent of the population variance. The first of these is "Student's" $t=m\sqrt{n}/s=\sqrt{n}(m/\sigma)/(s/\sigma)$, which has the requisite property. We know that m is normally distributed with standard deviation $\sigma/\sqrt{n}$, and that $s^2=\sum(x-m)^2/(n-1)$ is the best estimate of σ^2 in samples of n. For large samples, a consistent t tends therefore to normality with unit variance; its chief importance, however, lies in its use for small samples.

First, in standard variables, $m\sqrt{n}$ is the distance from the origin to the plane $\sum x_i=mn$. Moreover, $r_{n-1}=s\sqrt{n-1}$ is the distance within the hyperplane from the foot of the perpendicular, as before. Hence, the ratio $m\sqrt{n}/r_{n-1}=t/\sqrt{n-1}$ is the cotangent of the angle made by the radius vector from the origin to a point x of n-space with the normal to a family of parallel hyperplanes. Taking this normal as one axis, the distance from the origin may be integrated out over thin conical shells of angle θ with this direction. In one dimension, two lines can only make the angle 0 or π. In two dimensions, the thin sector has the "volume" element for the "cone" $2rd\theta dr$. In three dimensions, we apply the theorems of Pappus to get the volume element $2\pi r^2 \sin\theta dr d\theta$ over the conical shell, by simple extension to n dimensions, $(n-1)\,(n-2)\,cr^{n-1}\sin^{n-2}\theta dr d\theta$. Integrating out the r, the elementary probability is seen to be $c\sin^{n-2}\theta d\theta$, where c is hereafter treated throughout as a generic constant to be determined at the last step by equating the total probability to unity. Putting $u=\cot\theta$, dP transforms to $c(1+u^2)^{-n-2/2}(1+u^2)^{-1}du$. Finally, $u=t/\sqrt{n-1}$, which gives

THEOREM 5. *The probability of* $m\sqrt{n(n-1)}/\sum[x_i-m]^2\leqq t$ *where* $m=\sum x_i/n$ *and the* x_i *are independent normal variables with zero population mean and an identical variance is*

$$(15)\qquad P=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{n-1}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{t}\left(1+\frac{t^2}{n-1}\right)^{-n/2}dt.$$

Suppose we have two independent random samples of n_1, n_2 members respectively from the same standard normal population. Since $m_1\sqrt{n_1}$, $m_2\sqrt{n_2}$ are distances, independent and normally distributed, Theorem 1 applies to prove that m_1-m_2 is also normal with variance $1/n_1+1/n_2$, hence $(m_1-m_2)/\sqrt{(1/n_1+1/n_2)}$ is a normally distributed variable in standard measure. From this, we get a cotangent provided we can divide by some r_k. This is best done by taking $r_k^2=r_{n_1-1}^2+r_{n_2-1}^2$, *i.e.*, by combining the distance in the remaining n_1+n_2-2 dimensions, orthogonal to both m_1 and m_2. This gives

THEOREM 6. *For two independent samples from the same normal population*

$$\frac{m_1 - m_2}{\sqrt{\dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{m_1 - m_2}{\sqrt{(n_1 - 1)s_1^2 + (n_2 - 1)s_1^2}}\sqrt{\frac{n_1 n_2(n_1 + n_2 - 2)}{n_1 + n_2}}$$

has the t distribution with n_1+n_2-2 degrees of freedom. If the mean from the first and the variance from the second are used, $t=m_1\sqrt{n_1}/s_2$ with n_2-1 degrees of freedom.

In the first portion, standard measure is unnecessary in view of the fact that the expression for t is independent of both μ and σ. The second part follows immediately upon consideration of the fact that the degrees of freedom were associated only with the estimate of variance; for, $m\sqrt{n}$ represents just one fixed direction, the movement over the "surface" of the cone being associated with $r_{n-1}^2=(n-1)s^2$ which gives the degrees of freedom in t.

6. Distribution of the variance ratio. A most important property of r_n^2 is that it may be broken up into components of the same type. Each component divided by its degrees of freedom gives an estimate of σ^2. The sum of squares is the only analytic non-negative function of the coordinates that may thus be resolved into additive components without change of form. This simple algebraic fact is the basis, for example, of dynamical theorems like those of Bertrand and Kelvin on the energy of a system after a certain number of constraints. In statistics, the use made of this resolution into components is called analysis of variance, and consists of testing the above independent estimates of σ^2 for compatibility. From the preceding sections, it is clear that the ratio of any two such estimates is independent of both the population parameters μ and σ^2.

Assuming therefore a normal standard population without loss of generality, $s_1^2/s_2^2=(n_2-1)r_{n_1-1}^2/(n_1-1)r_{n_2-1}^2$. Now r_{n_1-1}/r_{n_2-1} is again the cotangent (or tangent) of an angle with this difference: that in the t distribution, the numerator was confined to one fixed direction by taking $n_1=1$ whereas it is now allowed to move freely through n_1-1 dimensions independently of the denominator. Ac-

cordingly, the n_2 dimensions of the denominator contribute $c'' \sin^{n_2-2}\theta d\theta$ as before, which must be multiplied by $c' \cos^{n_1-2}\theta d\theta$ for the swing of r_{n_1-1} (at right angles). The total elementary probability is, therefore, $c \sin^{n_2-2}\theta \cos^{n_1-2}\theta d\theta$. With $u=\cot\theta$, we get

THEOREM 7. *In two independent random samples of size n_1, n_2 from the same normal population, the distribution of $F=s_1^2/s_2^2$ is given by*

$$P(F \leqq t) = \frac{\Gamma\left(\frac{n_1+n_2-2}{2}\right)(n_1-1)^{(n_1-1)/2}(n_2-1)^{(n_2-1)/2}}{\Gamma\left(\frac{n_1-1}{2}\right)\Gamma\left(\frac{n_2-1}{2}\right)} \cdot \int_0^t \frac{F^{(n_1-3)/2}dF}{[(n_2-1)+(n_1-1)F]^{(n_1+n_2-2)/2}}. \tag{16}$$

The distribution of $z=\log\sqrt{F}$ has special advantages over that of F in the way of asymptotic formulae, and is therefore commonly used for accurate work. In practice, s_1^2, s_2^2 are so labelled as to give $F \geqq 1$, $z \geqq 0$.

7. Distribution of the coefficient of correlation in samples from an uncorrelated normal universe. The product-moment correlation (briefly the correlation) coefficient is estimated from a random sample of n pairs $(x_1, y_1) \cdots (x_n, y_n)$ by

$$r = \sum (x_i - \bar{x})(y_i - \bar{y}) / \sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}, \tag{17}$$

$$\bar{x} = \sum x_i/n \qquad \bar{y} = \sum y_i/n.$$

The basic assumption is that the coordinate pairs are sampled from an uncorrelated bivariate normal population, the elementary probability being $2\pi \exp -(x^2-y^2)/2$, using standard measure without loss of generality because r as defined above does not depend upon the population parameters.

In this case, we superpose the y-space upon the x-space by rotation, much as two pictures on transparent film may be superposed upon a screen. In this, the x_i and the y_i axis are to coincide for each i, which does not imply any relationship between the generic points x and y. Then, clearly, $r=\cos\theta$, θ being the angle between the two (independent) directions from the points $\bar{x}$, $\bar{y}$ to the points x, y respectively. The family of hyperplanes $\bar{x}$-const. coincides with the family $\bar{y}=$const. so that we can rotate both spaces simultaneously without destroying the original correspondence, and have only to investigate the distribution of $\cos\theta$ in the superposed $n-1$-space within the hyperplanes.

Here, the x-radius vector traverses the whole space without restriction, hence may be integrated out, the y-vector has only the restriction of making the angle θ with the first, so that its radial distance may be integrated out also over thin conical shells as in the two preceding distributions. This gives the total elementary probability $c \sin^{n-3}\theta d\theta$, the exponent being 2 less than the dimensions

$n-1$ of the free y-space (under the restriction $\bar{y}=$const.). This gives for the distribution of r,

$$P(r \leqq R) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)} \int_{-1}^{R} (1-r^2)^{(n-4)/2} dr. \tag{18}$$

Our method of derivation shows the intimate relation between t and F, which is that $t=\sqrt{F}$ essentially when $n_1=2$. Moreover, the relation between r and t is also immediately evident, given that $r=\cos\theta$ with $t=\sqrt{(n-1)}\cdot\cot\theta$ for a sample of n. Seeing that for t we have one more dimension at the start than for r when generating the hypercone of angle θ, it follows that $t=r\sqrt{(n-2)}/\sqrt{(1-r^2)}$ has the Student distribution with $n-2$ degrees of freedom.

MORE MODIFIED SERIES

J. W. BRADSHAW, University of Michigan

1. **Introduction.** In a recent article* the author pointed out that in certain cases the remainder of an infinite series, after summing a certain number of terms, can conveniently be expressed as an infinite continued fraction. We recall, with a slight change of notation, the two formulas there obtained as illustrations:

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = \sum_{n=1}^{k} (-1)^{n-1}\frac{1}{n} + (-1)^k \Big/ \left\{2k+1+\mathop{K}_{s=1}^{\infty}\frac{s^2}{2k+1}\right\},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{k} \frac{1}{n^2} + 2 \Big/ \left\{2k+1+\mathop{K}_{s=1}^{\infty}\frac{s^4}{(2s+1)(2k+1)}\right\}.$$

As further illustrations of results obtained by the method, we cite similar formulas for the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-2}$ and $\sum_{n=1}^{\infty}n^{-3}$ and a formula, obtained from the former, for $\sum_{n=1}^{\infty}n^{-2}$. Since the derivation and proof of the formulas of the earlier article are there given in some detail, and since the proofs of these new formulas can be carried through on similar lines, the formulas are stated without proof.

Attention may be called to a marked contrast between the continued fractions of the earlier formulas and those here involved, in that while the denominators in the former are linear in k and the numerators are positive, here the denominators are quadratic in k and the numerators are negative. This has an important bearing on the estimate of error involved in using a convergent; in the former case consecutive convergents are on opposite sides of the limit, while now all convergents are on the same side.

* J. W. Bradshaw, Modified series, this MONTHLY, vol. 46, 1939, pp. 486–492.

In this paper we call attention to the rapidity of convergence of the continued fractions involved to show that the method furnishes a really powerful tool for numerical calculation, powerful enough to give a feasible means of checking Stieltjes' calculation of $\sum n^{-2}$ and $\sum n^{-3}$, which presumably was based on the Euler-Maclaurin Sum Formula.

2. The three formulas. The formulas which give the remainder in the case of the three series mentioned are

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}=\sum_{n=1}^{k}(-1)^{n-1}\frac{1}{n^2}+\frac{(-1)^k}{2}\Big/\left\{k^2+k+1+\mathop{K}_{s=1}^{\infty}\frac{-s^4}{s^2+(s+1)^2+k^2+k}\right\}, \tag{1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{k}\frac{1}{n^3}+1\Big/\left\{2k^2+2k+1+\mathop{K}_{s=1}^{\infty}\frac{-s^6}{s^3+(s+1)^3+(2s+1)(2k^2+2k)}\right\}, \tag{2}$$

and then, inasmuch as $\sum_{n=1}^{\infty}n^{-2}=2\sum_{n=1}^{\infty}(-1)^{n-1}n^{-2}$ also

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=2\sum_{n=1}^{k}(-1)^{n-1}\frac{1}{n^2}+(-1)^k\Big/\left\{k^2+k+1+\mathop{K}_{s=1}^{\infty}\frac{-s^4}{s^2+(s+1)^2+k^2+k}\right\}. \tag{3}$$

In these formulas k is an arbitrary positive integer, the number of terms of the series summed directly. It may be of interest to remark that, if by $\sum_{n=1}^{0}$ we understand that no terms of the series are taken, the formulas reduce for $k=0$ to Euler's well known theorem of the equivalence of a series and a continued fraction.

3. Check of two of Stieltjes' values. Stieltjes in his table* of values of $S_k=\sum_{n=1}^{\infty}n^{-k}$ gives

$$S_2=1.64493\ 40668\ 48226\ 43647\ 24151\ 66646\ 03,$$
$$S_3=1.20205\ 69031\ 59594\ 28539\ 97381\ 61511\ 46,$$

to 32 decimal places, though he claims accuracy to only 30 places. To check the first of these values we take $k=40$ in formula (3) and compute $2\sum_{n=1}^{40}(-1)^{n-1}n^{-2}$ to 35 decimal places

$$1.64432\ 46821\ 00847\ 49998\ 87572\ 60979\ 57757.$$

* T. J. Stieltjes, Table des valeurs des sommes $S_k=\sum_{n=1}^{\infty}n^{-k}$, Acta Mathematica, vol. 10, 1887, pp. 299–302. See also J. W. L. Glaisher, Tables of $1\pm2^{-n}\pm3^{-n}\pm4^{-n}\pm$ *etc.* and $1+3^{-n}+5^{-n}+7^{-n}+$ *etc.* to 32 places of decimals, Quarterly Journal of Mathematics, vol. 45, 1914, pp. 141–158.

To this we add the reciprocal B_s/A_s of a convergent of the continued fraction

$$1641 - \frac{1\ |}{|\ 1645} - \frac{16\ |}{|\ 1653} - \frac{81\ |}{|\ 1665} - \frac{256\ |}{|\ 1681} - \frac{625\ |}{|\ 1701} - \frac{1296\ |}{|\ 1725} - \frac{2401\ |}{|\ 1753} \cdots .$$

Now it can be shown that the error committed by using for the infinite continued fraction a particular convergent A_s/B_s is less than $2(s+2)$ times the difference between the reciprocal of this convergent and the reciprocal of the next. Computing B_6/A_6 and B_7/A_7 to 35 places, we find that the difference is less than 3 in the 34th place, and hence B_6/A_6 yields a result correct to 32 places. The value obtained by this method agrees even in the 32d place with that given by Stieltjes. Though A_7 and B_7 consist of 26 and 23 figures, respectively, the value of B_7/A_7 to 35 decimal places can be obtained by means of a calculating machine without excessive labor.

For a similar calculation with formula (2) a smaller value of k will answer the purpose. We may for S_3 take $k=20$ and again use B_6/A_6; we obtain a result which is smaller than Stieltjes' value by one unit in the 32d place. Formula (2) furnishes an easy verification of Kummer's remark* that direct summation would require 10,000,000 terms of $\sum_{n=1}^{\infty} n^{-3}$ to give a result correct to 14 places. We have only to put the error, 5 in the 15th place, equal to B_0/A_0 to determine the required value of k.

4. Generalizations. As in the illustrations given in the earlier article, generalizations of these formulas are readily derived:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(nx+y)^2} = \sum_{n=1}^{k} (-1)^{n-1} \frac{1}{(nx+y)^2} + \frac{(-1)^k}{2} \Big/ \left\{ (kx+y)^2 + x(kx+y) + x^2 + \mathop{K}_{s=1}^{\infty} \frac{-x^4 s^4}{(kx+y)^2 + x(kx+y) + x^2(2s^2+2s+1)} \right\}, \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{(nx+y)^3} = \sum_{n=1}^{k} \frac{1}{(nx+y)^3} + \frac{1}{x} \Big/ \left\{ 2(kx+y)^2 + 2x(kx+y) + x^2 + \mathop{K}_{s=1}^{\infty} \frac{-x^4 s^6}{(2s+1)[2(kx+y)^2 + 2x(kx+y) + x^2(s^2+s+1)]} \right\}, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{1}{(nx+y)^2} = 2\sum_{n=1}^{k} (-1)^{n-1} \frac{1}{(nx+y)^2} + (-1)^k \Big/ \left\{ (kx+y)^2 + x(kx+y) + x^2 + \mathop{K}_{s=1}^{\infty} \frac{-x^4 s^4}{(kx+y)^2 + x(kx+y) + x^2(2s^2+2s+1)} \right\}. \tag{6}$$

* E. E. Kummer, Eine neue Methode die numerischen Summen langsam convergierender Reihen zu berechnen, Journal für Mathematik, vol. 16, 1837, pp. 206–214.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE TEACHING OF DIFFERENTIAL EQUATIONS

KARL MENGER, University of Notre Dame

The purpose of this note is the exposition of a well-known fact of physics which might be used with benefit in teaching differential equations, especially, in helping students to visualize them. In solving the equations we shall point out minor shortcomings, from the logical point of view, of the traditional exposition in textbooks, and indicate how they should be corrected.

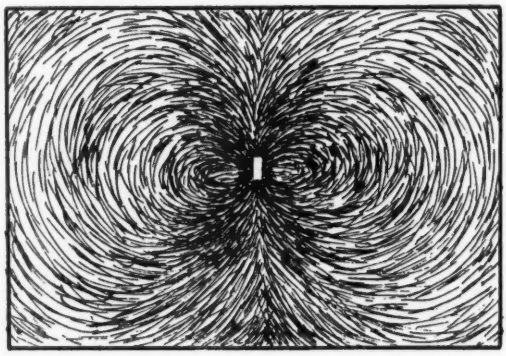


FIG. 1

The simplest introduction to the theory of differential equations is offered by nature. Cover a thin bar magnet with a horizontal piece of cardboard, and sprinkle fine iron splinters over the latter. Each splinter after coming to rest will assume a definite direction, depending upon its location. Before us is displayed the direction field associated with a first order differential equation. The splinters fit together along lines, called the lines of force of the magnet in the plane of the cardboard. These lines represent the solutions of the differential equation.

If the magnet covered by the horizontal cardboard is in a vertical position touching the cardboard at one pole, then each splinter will point toward the pole. If we choose this pole as the origin of a cartesian coördinate system, the splinter at the point (x, y) thus has the slope y/x. Hence the differential equation associated with the magnetic direction field reads $y'(x) = y(x)/x$ for $x \neq 0$. The splinters fit together along the straight lines through the origin. With the excep-

tion of the Y-axis,* these lines $y=Cx$ represent solutions $y(x)=Cx$ of the differential equation. The simplest correct way of showing that the equation has no other solution† seems to be by treating it as a Clairaut equation,‡ that is to say, by concluding that for each solution $y(x)$ of the equation and for each $x\neq 0$ we have

$$y''(x)=\left(\frac{y(x)}{x}\right)'=\frac{xy'(x)-y(x)}{x^2}=\frac{x\dfrac{y(x)}{x}-y(x)}{x^2}=0$$

and consequently $y(x)=Cx+D$ where D obviously must equal zero.

If the magnet, NS, is horizontal (see Fig. 1), then in the plane of the cardboard we introduce 1) a cartesian coördinate system whose origin is at the center of the magnet and whose Y-axis passes through the poles N and S, and 2) a polar coördinate system whose axis coincides with OX. A splinter at the point $P=(x,\ y)=(r,\ \theta)$ is attracted by one pole, say, N and repulsed by the other (see Fig. 2). The angle ϕ between the resulting direction of the splinter and the radius vector OP satisfies the approximate equality§

$$\tan\phi=\tfrac{1}{2}\cot\theta \tag{1}$$

provided that the length $NS=2l$ of the magnet is small compared with $OP=r$. Since $\tan\theta=y/x$, we have $\tan\phi=x/2y$ and for the angle $\theta-\phi$ between the splinter and the positive X-axis

$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{2}{3}\frac{y}{x}-\frac{1}{3}\frac{x}{y}\quad\text{for}\quad x\neq 0\quad\text{and}\quad y\neq 0.$$

The differential equation associated with this magnetic direction field is thus‖

$$y'(x)=\frac{2}{3}\frac{y(x)}{x}-\frac{1}{3}\frac{x}{y(x)}. \tag{2}$$

* The Y-axis cannot be represented by any function $y(x)$. By choosing another cartesian coördinate system with the origin O we see that the former Y-axis represents a line of force.

† More precisely: no other solution *which is differentiable even for* $x=0$, though the differential equation is only defined for $x\neq 0$. Indeed, $y(x)=|x|$ is an example of a function which is not of the form Cx and yet is a solution of the differential equation where the latter is defined. The general solution in the latter sense is $y(x)=Cx+D|x|$.

‡ See the author's article, Differential Equations, in vol. 2 of Practical Mathematics, New York City, 1943.

The reasoning that $y'(x)/y(x)=1/x$ by integration implies $\log|y(x)|=\log|x|+c$ and hence $y(x)=Cx$, for some $C\neq 0$, is valid only for values of x for which $y(x)\neq 0$, since $y(x)$ appears in a denominator. All we can conclude in this way is that the solutions which for every $x\neq 0$ assume values $\neq 0$, are of the form Cx, $(C\neq 0)$. This reasoning must be supplemented by the slightly deeper remark that a solution which for one $x_0\neq 0$ assumes the value 0, is necessarily equal zero for each x. This function happens to be of the form Cx for $C=0$.

§ See, *e.g.*, S. G. Starling, Electricity and Magnetism, 1927, p. 17 sq.

‖ See, *e.g.*, Watson and Burbury, The Mathematical Theory of Electricity and Magnetism, vol. 1, 1885, p. 267.

Introducing the auxiliary function* $z(x)=y(x)/x$ we obtain

$$xz'(x)+z(x)=\frac{2}{3}z(x)-\frac{1}{3}\frac{1}{z(x)},$$

thus

$$z'(x)=-\frac{1}{3x}\left(z(x)+\frac{1}{z(x)}\right) \quad \text{and} \quad \frac{z(x)}{z^2(x)+1}\cdot z'(x)=-\frac{1}{3x}.$$

By integration of these two equal functions of x, one of which is free of $z(x)$ while the other one is the product of $z'(x)$ and a function of $z(x)$, we find†

$$\tfrac{1}{2}\log(z^2(x)+1)=\log\left|x^{-1/3}\right|+\text{const.} \quad \text{and} \quad z^2(x)+1=Cx^{-2/3}$$

for some constant C which obviously must be positive. For the functions $y(x)$ in which we are interested we thus obtain

$$y^2(x)+x^2=Cx^{4/3} \quad \text{and} \quad (y^2(x)+x^2)^3=Cx^4 \text{ for some } C>0,$$

hence the following explicit expressions

$$y(x)=+\sqrt{Cx^{4/3}-x^2} \quad \text{and} \quad y(x)=-\sqrt{Cx^{4/3}-x^2}$$

for some $C>0$ and $0\neq|x|\leqq C^{3/2}$.

The equations of these lines in our polar coördinate system are $r=C\cos^2\theta$. The equipotential lines of the magnet, *i.e.*, the orthogonal trajectories of the lines of force, satisfy the differential equation

$$y'(x)=-1\Big/\left(\frac{2}{3}\frac{y(x)}{x}-\frac{1}{3}\frac{x}{y(x)}\right)$$

which is homogeneous, too. The polar equations of the solutions are $r=C\sqrt{\sin\theta}$.

For the sake of completeness a short proof of the approximate equality (1) may be added (see Fig. 3). The only assumption is the general attraction law. According to it, the vectors PQ and PQ' representing the two forces acting on the point P are proportional to the reciprocal squares of the distances from P to N and S, respectively, that is to say,

$$PQ=\frac{k}{r^2+l^2-2rl\sin\theta} \quad \text{and} \quad PQ'=QR=\frac{k}{r^2+l^2+2rl\sin\theta}.$$

If we set

* It seems to be about time to replace the terms "new variable" and "separation of variables" by expressions more adequate from the point of view of logic. (*Cf.* the author's article, Differential equations, mentioned in 3.)

† In order to avoid the inaccuracy criticized before, we may supplement the above reasoning by a remark taking into account that $z'(x)$ is undefined for values of x for which $z(x)=0$. From Rolle's theorem for functions which are differentiable in the interior of an interval and only continuous at its endpoints, we infer: No solution $z(x)$ can assume the value 0 at two different positive or two different negative values of x, since $z'(x)\neq 0$ for $x\neq 0$. It follows that also each solution $y(x)$ of (2) has at most one positive and one negative zero. The explicit expression of these solutions shows that each of them has exactly one positive and one negative zero, *viz.*, $\pm\sqrt{C^3}$.

$$\alpha = \sphericalangle RPQ, \qquad \beta = \sphericalangle QRP, \qquad \omega = \sphericalangle PQR = \sphericalangle NPS,$$

the tangent law yields

$$\tan\frac{\beta-\alpha}{2}:\tan\frac{\beta+\alpha}{2} = (PQ - RQ):(PQ + RQ),$$

hence

$$\tan\frac{\beta-\alpha}{2} = \cot\frac{\omega}{2}\cdot\frac{PQ - RQ}{PQ + RQ}.$$

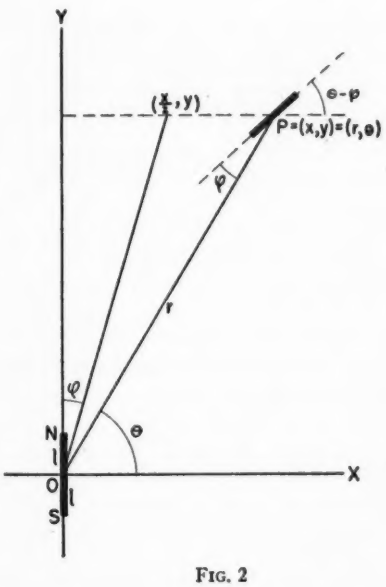


FIG. 2

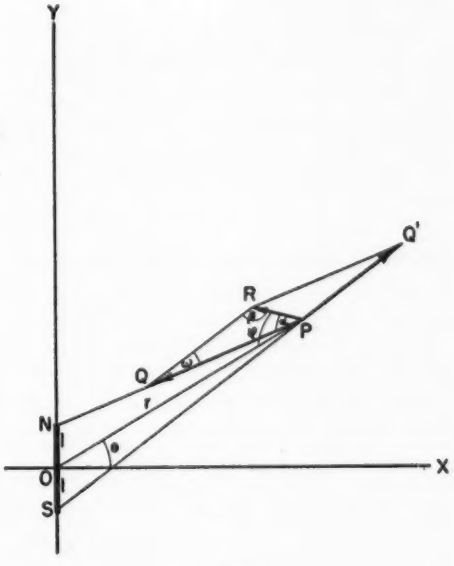


FIG. 3

If l is small compared with r, then ω is small compared with 90° and

$$\cot\frac{\omega}{2} \sim \csc\frac{\omega}{2} \sim \frac{r}{l\cos\theta} \quad \text{and} \quad \frac{PQ - RQ}{PQ + RQ} \sim \frac{4kl\sin\theta/r^3}{2k/r^3} = \frac{2l\sin\theta}{r}.$$

Hence

$$\tan\frac{\beta-\alpha}{2} \sim 2\tan\theta.$$

Moreover $\phi \sim \alpha \sim 90° - (\beta-\alpha)/2$. Thus

$$\tan\phi \sim \cot\frac{\beta-\alpha}{2} \sim \frac{1}{2}\cot\theta.$$

MORE GEOGRAPHICAL QUESTIONS

L. R. FORD, Illinois Institute of Technology

The geographical question propounded by Professor E. J. Moulton in the April MONTHLY suggests others.* If a man walks ten miles due south, then ten miles due east, then ten miles due north, then ten miles due west, he will ordinarily not be at the starting point. If his point of departure P is on the campus of Illinois Institute of Technology, he will find himself 114 feet west of P and with wet feet.

What is the locus of P in order that the man shall terminate his journey exactly at the starting point?

One obvious possible position of P is anywhere on the parallel of latitude 5 miles north of the equator. But there are many other possibilities. Let θ be the latitude of P counted positive to the north and negative to the south, and measured in radians; and let s (in this case 10 miles) be the length of each leg of his journey. Let R be the radius of the earth in miles.

The second part of his journey lies along a parallel of latitude θ', where

$$\theta' = \theta - s/R.$$

The radii of the two parallels involved are

$$r = R\cos\theta, \qquad r' = R\cos\theta'.$$

The angles subtended at the centers of these circles by an arc of length s are s/r and s/r'. The end of the journey will fall on P if these subtended angles are equal or if they differ by a multiple of 2π. We thus have

$$\frac{s}{R\cos\theta} - \frac{s}{R\cos\theta'} = 2n\pi,$$

or

$$\sec\theta - \sec\theta' = 2n\pi R/s,$$

where $n = 0, \pm 1, \pm 2, \cdots$. We shall show that this equation has exactly one solution for each value of n.

We shall show that

$$f(\theta) = \sec\theta - \sec\theta' = a$$

has exactly one solution in the interval to which θ is restricted,

$$-\frac{\pi}{2} + \frac{s}{R} < \theta < \frac{\pi}{2}.$$

As θ traces this interval from left to right $f(\theta)$ varies from $-\infty$ to $+\infty$. It thus

* Professor F. A. Foraker has called my attention to his interesting list of geographical questions in Education, vol. 38, 1917, pp. 157–159. Professor Ford's first question is essentially problem 13 of this list. M. J. W.

takes on any real value a. If a were taken on more than once, we should have

$$f'(\theta) = \sec\theta \tan\theta - \sec\theta' \tan\theta' = 0$$

somewhere in the interval, which we shall show to be impossible. From this

$$\sec^2\theta \tan^2\theta = \sec^2\theta' \tan^2\theta',$$

whence

$$(\tan^2\theta - \tan^2\theta')(1 + \tan^2\theta + \tan^2\theta') = 0.$$

This holds only if $\theta=\theta'$, which is untrue, or $\theta=-\theta'$. The latter is an extraneous root, since it gives $f'(\theta)\neq 0$. There is thus, exactly one solution of the equation.

For $n=0$, we have the solution $\theta=s/2R=-\theta'$, which gives for P, the parallel of the latitude mentioned at the beginning. For $n=1, 2, 3, \cdots$ we have a sequence of parallels converging to the north pole, on any one of which P may lie. For $n=-1, -2, -3, \cdots$ we have a sequence of parallels converging to the parallel of latitude s miles from the south pole. The north pole and this latter limiting circle are excluded from the locus.

We consider also the following problem.

Pedro's banana farm is bounded by two east-west fences exactly two miles long and two north-south fences exactly one mile long. José's farm is bounded by east-west fences exactly one mile long and by north-south fences exactly two miles long. Which has the larger farm?

It is clear that each farm is bisected by the equator and that the area of each is somewhat in excess of two square miles. Let us find the area A of a "rectangular" farm extending s miles on each side of the equator and bounded by east-west fences w miles long.

The latitudes of the latter fences are $\pm s/R$ radians. The height of the zone around the earth between these parallels is $2R \sin (s/R)$ and the area of the whole zone is

$$Z = 2\pi R \cdot 2R \sin (s/R).$$

The radius of the parallel of latitude is $r=R\cos(s/R)$ and its circumference is $2\pi R\cos(s/R)$. The required area is to Z as w is to this circumference, whence

$$A = \frac{wZ}{2\pi R\cos(s/R)} = 2Rw\tan\frac{s}{R}.$$

The function $\tan(s/R)$ may be expanded in the usual power series for all s allowable by the problem; and the convergence is rapid for s/R small. We have

$$A = 2Rw\left(\frac{s}{R} + \frac{s^3}{3R^3} + \cdots\right) = 2ws + \frac{2ws^3}{3R^2} + \cdots.$$

For Pedro's farm we have $s=\frac{1}{2}$, $w=2$, for José's, $s=1$, $w=1$. We have then respectively

$$A_p = 2 + \frac{1}{6R^2} + \cdots, \qquad A_j = 2 + \frac{2}{3R^2} + \cdots.$$

We see that José's farm is the larger by $1/(2R^2)$ square miles, which is almost 125 squares inches.

This is the area of the two pages of the MONTHLY that the reader has open before him, exclusive of the top margins. Pedro's farm exceeds two square miles by one-third of this area.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

ON TRISECTING AN ANGLE

BARBARA STEINBERG, Student at Queen's College

There is a proof, or so I'm told
By scientists respected,
That certain angles stubbornly
Refuse to be trisected
With compass, straight edge, pen and ink.—
And that's the only ticket,
For Euclid says that other tools
Simply are not cricket.
Yet every year some hardy souls
Will publish refutations
Of scientific treatises
And learned dissertations.
They send out methods by the score
And hope that by insistence
They will succeed in lowering
The scientists' resistance.
Some hint that mathematicians
(Ugly insinuation)
Claim no solution's possible
Out of sheer desperation.
You think you've found a method, eh,
Which cannot be refuted?—
I tell you, sir, I know it's right!
Aha, but is it Euclid?

CLUB REPORTS 1943-44

Harvard Mathematical Club

A distinctly increasing spirit of cooperation with neighboring institutions was in evidence in 1943–44. Members of the Mathematics Clubs and Faculties of Greater Boston colleges, including those of Boston University, M.I.T., Regis, and Tufts, were frequent and welcome visitors. And, breaking all traditions, students of Radcliffe College are now admitted to active membership on an equal basis with Harvard. During the year the following talks were presented:

Gödel's theorems, by Dr. Lynn H. Loomis

Symbolic probability, by Dr. Irving Kaplansky

Curves without tangents, by Dr. Ralph Boas, Jr.

Picturing fractions, by Peter Frank

Two hundred and fifty years of hydrodynamic theory, by Professor Garrett Birkhoff

1152 Varieties of algebra, by Professor R. M. Frye (Boston University)

Through the window, by Professor D. V. Widder

Mathematical economics, by Professor W. W. Leontief

Polyhedra, by R. C. Buck (first prize winner)

Philosophy for the scientist, by Professor Philip Frank

Mathematics in South America, by Professor M. H. Stone

The trisection of the angle, by Dr. Irvin Cohen

Tracking the quadratic to its native lair, by Professor Robert Bruce (Boston University)

Equations with too many solutions, by Mrs. Ralph Boas, Jr. (Tufts)

Foundations of mathematics, by P. R. Masani

Bernouillian numbers, by George Williams (second prize winner)

Some applications of Boolean algebra, by Robert Hoskins

Geometry of the triangle and two-dimensional relativity, by Professor G. D. Birkhoff

Recipients of the Robert Fletcher Rogers Prizes for the best student talks were Mr. Buck ($35) and Mr. Williams ($15). Officers for 1943–44 were: President, Robert Hoskins; Vice-Presidents, Daniel Gorenstein and Richard Arens; Secretary, Joseph Zilber; Treasurer, George Mostow; Faculty Advisers, Dr. George Mackey and Dr. Irving Kaplansky.

Rho Theta, Saint Louis University

The first of the monthly meetings of *Rho Theta* for the year 1943–44 was held in the fall. After the election of officers, plans for future meetings were discussed and committees appointed. It was decided to have guest speakers to talk on topics pertinent to mathematics, and seven lectures were arranged. Among these were:

Probability, by Professor Francis Regan

Seismology, by E. J. Walker, Instructor in Geophysics

The method of least squares, by Professor Regan

Simplifying production mathematics, by W. J. Biermann, Graduate Fellow in Chemistry

Curve-fitting, by Professor Regan

The application of the Clausius-Clapeyron equation to an earth problem, by Albert Frank, Instructor in Geology.

Round table discussions were held during the year to decide in which branch of the service the members would be able to use their mathematics more readily, and several problems were presented to the group by the members and solved at subsequent meetings. Two banquets were held, the latter being the final meeting, which was concluded by a discussion on the subject

Mathematics and the war, by Professor A. E. Ross of the Department of Mathematics.

Officers for the fall term were: President, Joseph Heithaus; Vice-President, Edwin Mertz; Secretary-Treasurer, Herbert Gebhardt. Officers for the spring term were: President, John Quinn; Vice-President, Herbert Gebhardt; Secretary-Treasurer, Helen Ann Jackson.

Square Circle Club, Woman's College of the University of North Carolina

Due to the accelerated program of education, the meetings of the *Square Circle Club* were decreased to two for each semester, in addition to the annual initiation meeting for the induction of new members.

Wartime mathematics was the theme for the year, and the program included the reading of letters from mathematics graduates who are now serving in the armed forces of our country or working in wartime industries.

Use of the slide rule was the subject selected by the Juniors, and slide rules were passed out to the club members so that they could practice various calculations.

Navigational stars was the subject of a dialogue given by two Seniors at the initiation meeting in February. The interest thus aroused in astronomy led the Sophomores to schedule for their meeting a lantern slide lecture entitled

The solar system, by Miss Cornelia Strong, the Faculty Adviser. This was followed by a naked-eye study of the constellations from the roof.

Radar and the range finder was the theme for the Freshmen program, which emphasized the importance of radar as a weapon in this war.

The major social event of the year was our formal initiation of the largest group of new members in the history of the club. After several of our meetings informal social hours were held during which mathematical games were enjoyed and refreshments were served. Officers for the year 1943–44 were: President, Janet Hubbard; First Vice-President, Lucy Taylor; Second Vice-President, Kathleen Wicker; Secretary-Treasurer, Sue McGee; Social Chairman, Elaine Atkin; Faculty Adviser, Miss Cornelia Strong. Officers elected for 1944–45 were: President, Katherine Simpson; First Vice-President, Sue McGee; Second Vice-President, Eleanor Holmstine; Secretary-Treasurer, Lois Russell; Social Chairman, Hilda Mattox.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics for Navigators. By Delwyn Hyatt and B. M. Dodson. New York and London, McGraw-Hill Book Company, Inc., 1944. 7+106 pages. $1.25.

This book is intended primarily for those studying navigation "on their own." It is restricted to the mathematics of the subject with chapters on arithmetic, algebra, logarithms, plane geometry, plane trigonometry, spherical trigonometry, and oblique spherical triangles.

The first chapter deals only with arithmetical operations useful in navigation. Left-to-right addition and subtraction, and interpolation are used as unifying agents from the first and weld a number of isolated topics such as the sexagesimal systems, conversion of arc into time, and the finding of Greenwich hour angles from the *Nautical Almanac* into an interesting whole.

Approximate solution of spherical triangles by construction is outstanding and very useful, while the projection of the celestial triangle onto the plane of the meridian results in a very clear derivation of the formulas of Ageton's method, H. O. 211.

The compression of so many topics into such a little volume leads to occasional errors of omission and to inaccuracies of statement. These are perhaps most noticeable in the chapter on algebra, where a great many processes must be treated. Among them are the use of negative, fractional, and zero exponents after definition has been made for the positive integral case only; treatment of "cross-multiplying" in a proportion as though it would apply to any equation containing fractions; and the solution of linear equations in one unknown as though the method were generally applicable to equations of higher degree as well.

Two errors should perhaps be mentioned. On page 31, the following sentence might well be deleted: "In using base 10, the logarithm of 89, for example, means simply that 10 raised to the 'log of the 89th power' equals 89." And on page 60, in discussing the ambiguous case of the plane triangle with a, b, and A given, the authors write: "In actual practice there are two solutions when two sides and the angle opposite one of them are given," ignoring the fact that if a is not less than b, there is but one solution.

The orderly tabulation of solutions and the use of the haversine law are strong points of the book. No tables are included, it being assumed that the student will have ready access to Part II of *Bowditch's Useful Tables*, to which careful references are given.

R. C. HUFFER

Mathematics for Mariners. By C. E. Dimick and C. C. Hurd. New York, D. Van Nostrand Company, Inc., 1943. 7+182+69 pages. $2.75.

The authors state in the preface: "It is believed that the book contains the basic mathematics necessary for the undertaking of plane navigation, simple gunnery, seamanship, the problem of maneuvering at sea, and elementary marine engineering." They also say: "It is not to be expected that all the book up to and including Chapter X can be completed in a four or five weeks course."

They begin with arithmetic, since they have found that officer candidates and cadets who fail, do so "simply through slowness and inaccuracy in numerical calculations." The authors stress checks in all computations and use an elaborate system of casting out 9's in addition and subtraction as well as multiplication.

Exponents and logarithms are discussed in Chapter II and use of tables in Chapter III. Examples of interpolation, including double interpolation, are given with brief sections of tables such as are to be used for future marine work. While the Bowditch tables are not mentioned, it seems likely that the authors have in mind such tables which are widely used in navigation.

Chapter IV on Equations takes up linear, quadratic, and simultaneous linear equations with examples of problems from various phases of military use. The next chapter entitled "Geometry" gives some familiar definitions and theorems without proofs. Conversion of degrees to radians with appropriate definitions is carefully discussed, as is the use of the *mil*, which the army has asked to be introduced into the mathematical texts.

In three chapters on trigonometry, the authors give the student enough experience to enable him to solve any plane triangle. The trigonometric functions are defined by the right triangle method for angles less than 90°. Then follows a derivation of functions of angles of all sizes. It seems queer that they use an example from plane sailing involving departure and difference of latitude which is less familiar to most teachers than the standard definitions from a circle with rotating radius. The method used for determining the signs of the functions in the four quadrants makes use of what many teachers might consider an artificial method, but of course it gives the same results as the standard method.

Since celestial navigation, which is so important now, is based on spherical trigonometry, it is surprising that spherical trigonometry is omitted entirely from this text. Few people seem to be aware of a simple method of deriving at once the three sets of equations used in solving a spherical triangle. This method is due to Chauvenet and is given in less than four pages in a *Text-book of Field Astronomy* by G. C. Comstock (1908). All needed formulae, including those for right spherical triangles, can be derived for a class in two hours.

The text concluded with two chapters of vector diagrams for the solution of problems involving speed of ships and current or wind drift. The use of the maneuvering board is introduced and ten sample compass roses (one-half standard scale) are included at the end for practice use. This section of the book is

well-done and introduces simply and satisfactorily certain practical problems which are stressed in courses of deck officer training in the Navy and Coast Guard.

It is quite obvious that the mathematics given in this text is necessary to the student of marine problems. Some of it should have been thoroughly taught by the 10th grade. Some high schools have good courses in trigonometry which have been popular during the last two years. This book belongs to the level of such high schools.

A word should be added about the tables. It is gratifying to find five-place tables when the modern tendency is to four-place tables to save expense of printing. We wish the first figures of the mantissas of the logarithms had been printed. Also that the tables of tangents and cotangents had been printed alongside the sines and cosines, instead of making two separate tables. Tables of natural functions apart from their logarithms are acceptable and quite usual.

On the whole the authors have done well what they set out to do and have put into one book the mathematics necessary for mariners. The book will make a valuable handbook for those who have gotten the practical science of navigating and have neglected the theoretical.

C. M. HUFFER

Table of Reciprocals of the Integers from 100,000 through 200,009. 8+201 pages. $4.00.

Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments. 38+410 pages. $5.00.

Table of the Bessel Functions $J_0(x)$ and $J_1(z)$ for Complex Arguments. 44+403 pages. $5.00.

All of the above tables were prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency, conducted under the sponsorship of the National Bureau of Standards, published by the Columbia University Press, New York, 1943.

The tabulated values are given (in order of mention) to 7 decimal places, 8 significant figures, 10 decimal places. The complex argument $\rho(\cos\phi+i\sin\phi)$ of the Bessel functions is tabulated for ρ from 0 to 10 in steps of .01 and for ϕ from 0° to 90° in steps of 5°. The prefaces contain valuable information on available literature in numerical tables, including lists of corrections and numerical errors carried over into quite recent publications. The present tables are the result of independent computations and re-computations. The cover size is 11″ by 8″, the photo-offset reproduction is immaculately distinct and clear.

M. A. SADOWSKY

NEW BOOKS RECEIVED

Aircraft Analytic Geometry Applied to Engineering, Lofting, and Tooling. By J. J. Apalategui and L. J. Adams. New York and London, McGraw-Hill Book Company, Inc., 1944. 17+285 pages. $3.00.

Air Navigation Made Easy. By James Naidich. New York and London, McGraw-Hill Book Company, Inc., 1944. 9+124 pages. $1.75.

Basic Marine Navigation. By B. J. Bok and Frances Wright. Boston, Houghton Mifflin Company, 1944. 8+422 pages. $4.50. Kit of Practice Materials, $1.70.

Basic Mathematics for Engineers. By P. G. Andres, H. J. Miser, and Haim Reingold. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1944. 10+726 pages. $4.00.

Celestial Navigation. A Problem Manual. By Walter Hadel. New York and London, McGraw-Hill Book Company, Inc., 1944. 13+261 pages. $2.50.

Exact Values of the First 200 Factorials. By H. S. Uhler. New Haven, Uhler, 1944. 22 pages. $0.80.

Fourier Series. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 38.) By G. H. Hardy and W. W. Rogosinski. New York, The Macmillan Company, 1944. 8+100 pages. $1.75.

General Mathematics in American Colleges. By K. E. Brown. New York, Teachers College, Columbia University, 1943. 4+167 pages. $2.35.

Marine and Air Navigation. By J. Q. Stewart and N. L. Pierce. Boston, Ginn and Company, 1944. 12+472 pages. $4.50.

Mathematical and Physical Principles of Engineering Analysis. By W. C. Johnson. New York and London, McGraw-Hill Book Company, Inc., 1944. 10+346 pages. $3.00.

Mathematics for Aircraft Engine Mechanics. By Harold Griffiths. New York and London, McGraw-Hill Book Company, Inc., 1944. 15+367 pages. $2.50.

Mathematics for Exterior Ballistics. By G. A. Bliss. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1944. 7+128 pages. $2.00.

Modern Operational Mathematics in Engineering. By R. V. Churchill. New York and London, McGraw-Hill Book Company, Inc., 1944. 10+306 pages. $3.50.

The Philosophy of Bertrand Russell. (The Library of Living Philosphers, vol. 5.) Edited by P. A. Schilpp. Evanston and Chicago, Northwestern University, 1944. 16+815 pages. $4.00.

Plane and Spherical Trigonometry. By H. P. Doole. New York, Thomas Y. Crowell Company, 1944. 8+183 pages. $1.75.

Sistemas de Ecuaciones Analiticas en Terminos Finitos, Diferenciales y en Derivadas Parciales. (Monografias publicadas por la Facultad de Ciencias Mathematicas, Fisico-Quimicas y Naturales, applicadas a la Industria, Universidad Nacional del Litoral, No. 1.) By Beppo Levi. Rosario, Argentina, 1944. 218 pages. $8.00.-m/n.

Spherical Trigonometry. By Aaron Freilich, H. H. Shanholt, and Joseph Seidlin. New York, Silver Burdett Company, 1943. 4+140 pages. $1.28.

Tables of Lagrangian Interpolation Coefficients. Prepared by the Mathematical Tables Project, conducted under the sponsorship of the National Bureau of Standards. New York, Columbia University Press, 1944. 36+392 pages. $5.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 631. *Proposed by J. A. Tierney, Norwich University, Northfield, Vt.*

Express the number 64, using two fours (and the operations of arithmetic).

E 632. *Proposed by V. Thébault, San Sebastián, Spain*

What is the smallest radix which admits a perfect cube of the form *abab*?

E 633. *Proposed by N. A. Court, University of Oklahoma*

Given a point M and four spheres (A), (B), (C), (D) whose centers form a tetrahedron, let MEE' be the transversal from M to the two opposite edges BC, DA, and let spheres (E) and (E') be constructed coaxal with the pairs of spheres (B), (C), and (D), (A). We have analogous spheres (F) and (F'), (G) and (G'). Show that the sphere (M) coaxal with (E), (E') is likewise coaxal with (F), (F') and with (G), (G').

E 634. *Proposed by H. S. Wall, Northwestern University*

Let a matrix be constructed according to the following rules:

$$a_{1,1} = 1, \qquad a_{p,q} = 0 \quad \text{if} \quad p < q,$$

$$a_{p,1} = a_{p-1,2}, \qquad (p = 2, 3, \cdots),$$

$$a_{p,q} = a_{p-1,q-1} + a_{p-1,q+1} \qquad (p, q = 2, 3, \cdots).$$

Show that the sum of the products of corresponding elements in any two rows is equal to an element in the first column, *viz.*,

$$\sum_{r=1}^{\infty} a_{p,r}a_{q,r} = a_{p+q-1,1} \qquad (p, q = 1, 2, \cdots).$$

E 635. *Proposed by R. A. Rosenbaum, U.S.N.R.*

Derive parametric equations for the involute of the involute . . . (n times) of a circle (with the same starting point for each process of unwinding).

Correction to E630[1944, 348]. *Proposed by R. A. Rosenbaum, U. S. N. R.*

For $s' = s + \pi d$ read $s' = s + 2\pi d$.

SOLUTIONS

The Reinforced Phalanx

E 596 [1943, 633]. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

A square phalanx of less than five million soldiers was reinforced by ten equal squares and then formed a square of over ten million which could be divided into 400 equal squares. The reinforced army was less than three times the original. How many soldiers were in each of the 400 squares?

Solution by Colin Blyth, Queen's University. Let Y, X, and z be the respective sides of the original square, of a reinforcing square, and of one of the resulting 400 squares. Then

$$10X^2 + Y^2 = 400z^2.$$

We see that Y, and therefore also X, must contain the factor 10. Setting $X = 10x_1$, $Y = 10y_1$, we have $10x_1^2 + y_1^2 = 4z^2$. Thus y_1, and therefore also x_1, must be even. Setting $x_1 = 2x$, $y_1 = 2y$, we have

$$10x^2 + y^2 = z^2, \tag{1}$$

where $x = X/20$ and $y = Y/20$. Also, the given restrictions imply

$$35 < x < 50, \qquad 91 < y < 112, \qquad 158 < z < 194.$$

Now, the general solution of (1) (*cf.* H. N. Wright, *Theory of Numbers*, p. 96) is

$$x = 2kmn, \qquad y = k \cdot |rm^2 - n^2|, \qquad z = k(rm^2 + n),$$

where k, m, n, r, s are positive integers, $rs = 10$, and

$$(m, n) = (r, s) = (r, n) = (s, m) = 1.$$

By testing the fifteen possible pairs m, n with $r = 10$, $s = 1$, and the eighteen possible pairs with $r = 5$, $s = 2$, we find that the only solution of (1) satisfying the given inequalities is

$$x = 36, \qquad y = 111, \qquad z = 159$$

(corresponding to $r = 5$, $s = 2$, $m = 3$, $n = 2$, $k = 3$). Thus the number of soldiers in each of the 400 squares is $z^2 = 25281$.

Also solved by E. M. Berry, W. E. Buker, H. D. Larsen, F. L. Miksa, E. P. Starke, Richard Woollett, and the proposer.

The Circular Baseball League

E 601 [1944, 46]. *Proposed by C. T. Tobin, St. Francis Xavier College, Antigonish, N. S.*

A baseball league is formed by seven teams belonging to towns located one mile apart around a circular railroad. In order that each team shall play each

other, three games are held every day for a week. No team plays in any town more than once, nor plays more than one game a day. In each town, there is no more than one game a day, and there are never games on two consecutive days. There are no home games. At night each team travels to the scene of the next day's game, by the shortest route along the circular railroad. (Trains run in both directions.) A team that is going to be idle on any day returns (or remains) home the preceding night. When a team has completed its schedule, it returns home that night. The first games are played on Sunday; so the travelling starts on Saturday night and is completed on the following Saturday night. The towns are 1, 2, 3, 4, 5, 6, 7, though not in that order. Teams 3, 5, 6 never travel more than 2 miles any night. Teams 3, 4, 7 all travel the same total number of miles during the schedule. Teams 4 and 6 play at town 5 on Friday, and on that day there is also a game played at town 1.

Solution by F. L. Miksa, Aurora, Illinois. The order of towns is

4, 5, 3, 2, 6, 1, 7.

For the schedule of games, see Fig. 1, which indicates that teams 2 and 5 play in town 4 on Sunday, teams 3 and 7 in town 5 on Monday, and so on. We see that team 1 travels $1+3+2+1+2+3+2=14$ miles altogether; team 2, 17 miles; team 5, 9 miles; team 6, 12 miles; while 3, 4, and 7 travel 13 miles each.

	4	5	3	2	6	1	7
Sun.	25			36		47	
Mon.		37			24		15
Tue.			27	14		56	
Wed.	67	12			35		
Thu.			16	57			34
Fri.		46			17	23	
Sat.	13		45				26

FIG. 1

	1	2	3	4	5	6	7
1	47			56	23		
2		45				37	16
3	26		57		14		
4		36		27		15	
5			12		67		34
6	35	17				24	
7			46	13			25

FIG. 2

Fig. 1 is a kind of modified Eulerian square: a distribution of the 21 pairs of the digits $1, \cdots, 7$ in suitable cells of a 7×7 square so that every row or column contains six of the seven digits. Each column is headed by the digit not contained in it (because there are no home games). Similarly, the digits missing from the respective rows are 1, 6, 3, 4, 2, 5, 7. Fig. 2 shows such a table in its standard form, with rows and columns permuted in such a way that the digit k is absent from the kth row and from the kth column. This suggests the new problem of enumerating such standard squares. To avoid too great multiplicity we may stipulate that the main diagonal (as in this instance) shall contain no empty cells.

A Corollary to E 552

E 604 [1944, 46]. *Proposed by Free Jamison, U. S. Navy Air Navigation School*

Is there any numerical solution of E 552 with $4<b<c<33800$?

Solution by E. P. Starke, Rutgers University. A negative answer to the question is indicated. Using the notation of Jamison's solution for E 552 [1943, 563], we have

$$c = mb, \qquad (a-1)^2 = nb, \qquad m^2b = n + a,$$

where n and b are perfect squares. With $n=s^2$, $b=y^2$, we have to consider $m^2y^2=s^2+sy+1$ or

$$x^2 - (4m^2+1)y^2 = -4, \tag{1}$$

where $x=2s+y$. Evidently x and y are both even or both odd.

In the latter case there is no solution of (1) with $m>1$. For, if $x=p$, $y=q$ is a solution of $x^2-Dy^2=\pm N$, $0<N<\sqrt{D}$, with p prime to q, then p/q is a convergent of the simple continued fraction expansion of $\sqrt{D}$. (See Wright, *Theory of Numbers*, New York, 1939, pp. 40–41.) But there is no convergent of

$$\sqrt{4m^2+1} = 2m + \cfrac{1}{4m + \cfrac{1}{4m + \cfrac{1}{4m + \cdots}}} \tag{2}$$

with both numerator and denominator odd.

Thus the x and y of (1) must be both even, say $x=2\xi$, $y=2\eta$, and we have to consider

$$\xi^2 - (4m^2+1)\eta^2 = -1. \tag{3}$$

The first three convergents of (2) are $2m$, $(8m^2+1)/4m$, $(32m^3+6m)/(16m^2+1)$. By the usual theory of the Pellian equation, every solution of

$$\xi^2 - (4m+1)\eta^2 = \pm 1$$

comes from a convergent ξ/η of (2). Since each period of the recurring continued fraction (2) contains a single quotient, every convergent provides a distinct solution, convergents of odd order giving the negative sign. Thus the two smallest solutions of (3) are

$$\xi = 2m, \quad \eta = 1; \qquad \xi = 32m^3 + 6m, \quad \eta = 16m^2 + 1.$$

As applied to (1), the first gives $b=4$; the second gives $b=4(16m^2+1)^2$, whence, if $m=c/b\geqq 2$, we have $b\geqq 16900$ and $c\geqq 33800$. Thus every solution lies outside the specified range.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4128. *Proposed by C. E. Springer, University of Oklahoma*

Consider the tangent planes to a sphere at three points A, B, C of a curve lying on the sphere. Let R be the limiting point of intersection of the planes as B and C move independently along the curve and approach coincidence with A. Each curve on the sphere through A has its corresponding R point. Prove that the curves through A, the locus of whose R points is a certain straight line lying in the tangent plane to the sphere at A, are the loxodromes through A.

4129. *Proposed by F. C. Gentry, Louisiana Polytechnic Institute*

Prove that the vertices of the original tetrahedron and those of either of the other tetrahedrons of a desmic system of tetrahedrons are the centers of the eight spheres which touch the faces of the third member of the system.

4130. *Proposed by J. R. Musselman, Western Reserve University*

Show that

$$\sum_{j=1}^{n} {}_nC_j \frac{(-1)^{j-1}}{j} = \sum_{j=1}^{n} \frac{1}{j}.$$

4131. *Proposed by V. Thébault, San Sebastián, Spain*

Let $ACBD$ be a skew quadrangle; planes perpendicular at A to AC, at C to CB, at B to BD, at D to DA form a tetrahedron $A_1C_1B_1D_1$ with the centroid G_1. Show that the powers of A with respect to the sphere (G_1C) on G_1C as diameter, of C with respect to (G_1B), of B with respect to (G_1D), of D with respect to (G_1A) are equal.

SOLUTIONS

Exponential Function as Quotient of Infinite Products

4072 [1943, 124]. *Proposed by Richard Bellman, Brooklyn College*

Show that

$$e^x = \frac{(1-x^2)^{1/2}(1-x^3)^{1/3}(1-x^5)^{1/5}\cdots}{(1-x)(1-x^8)^{1/6}(1-x^{10})^{1/10}\cdots}, \qquad |x| < 1,$$

where the exponents in the numerator are integers with an odd number of unrepeated prime factors; and those in the denominator have an even number of unrepeated prime factors.

Solution by R. C. Buck, Cambridge, Mass. Let us consider the function

$$f(x) = -\sum_{n=1}^{\infty} \frac{\mu(n) \log (1 - x^n)}{n}, \qquad |x| < 1$$

where $\mu(n)$ is the Möbius function. Then,

$$f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{v=1}^{\infty} \frac{x^{vn}}{v} \qquad \text{for} \quad |x| < 1$$
$$= \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \frac{\mu(n)}{nv} x^{nv}.$$

The coefficient of x^m in this expansion is

$$\sum_{n|m} \frac{\mu(n)}{m} = \frac{1}{m} \sum_{n|m} \mu(n) = 0 \quad \text{unless} \quad m = 1.$$

Hence, $f(x) = x$, and

$$e^x = \sum_{n=1}^{\infty} (1 - x^n)^{-\mu(n)/n},$$

which gives the desired result.

Equilateral Triangles on Sides of a Triangle

4076 [1943, 204]. *Proposed by Harold E. Gove, Major, Washington, D. C.*

Given the triangle $A_1A_2A_3$ show how to construct the triangle $B_1B_2B_3$ so that the triangles $B_jB_iA_k$ will be equilateral and exterior to $B_1B_2B_3$.

Solution by Howard D. Grossman, New York City. We may view the plane on that side on which B_1, B_2, B_3 appear in clockwise order. Then in order that $B_jB_iA_k$ lie outside $B_1B_2B_3$, it is necessary that B_j, B_i, A_k be also in clockwise order, i, j, and k being any permutation of the numbers 1, 2, 3 in cyclical order. This is equivalent in the complex plane to $B_j+\omega^2B_i+\omega A_k=0$, which uniquely determine $B_i=\frac{1}{2}(A_i-\omega A_j-\omega^2A_k)$, where $\omega=\cos 120°+i \sin 120°$. Geometrically this is equivalent to viewing the plane on that side on which A_1, A_2, A_3 appear in clockwise order, finding C_k on the same side of A_iA_j as A_k so that $A_iA_jC_k$ is equilateral and A_i, A_j, C_k are in clockwise order, and taking B_k as the midpoint of C_iC_j. This equivalence follows mechanically from $C_i+\omega^2A_j+\omega A_k=0$.

This however is not always sufficient, *e.g.*, an isosceles A-triangle of 120°, 30°, 30° or of $\theta(=\tan^{-1} 3\sqrt{3}$, about 79°), θ, $180°-2\theta$ gives a null B-triangle and an A-triangle of $120+°$, $30-°$, $30-°$ or of $\theta+$, $\theta+$, $(180°-2\theta)-$ gives a B-triangle overlapping one or more $B_jB_iA_k$-triangles. These cases have no solution.

Solved also by J. Rosenbaum.

Editorial Note. Rosenbaum gave the same method of locating the vertices B_i using geometrical considerations. He also gave remarks on the more difficult similar problem where the triangles $B_jB_iA_k$ are directly similar but not necessarily equilateral.

If $t_1, t_2, t_3, u_1, u_2, u_3$ are complex quantities for $A_1, A_2, A_3, B_1, B_2, B_3$, with the origin at any chosen point, $\lambda = e^{i\pi/3}$ then

$$u_1 = t_1 + \lambda t_2 + \lambda^{-1} t_3, \tag{1}$$

and u_2, u_3 are given by cyclic changes of subscripts. Since $\lambda + \lambda^{-1} = 1$ it is clear that the two triangles have the same centroid. If we set $t_3 = 0$, the equation (1) means that A_3A_2 is rotated through $+60°$ giving D_1 and the midpoint of A_1D_1 is B_1, similarly for D_2, B_2 and D_3, B_3, where D_1, D_2, D_3 are vertices of equilateral triangles constructed exteriorly on the sides of $A_1A_2A_3$. The line segments A_iD_i have equal lengths; for, if $A_1D_3A_3$ is rotated through $+60°$ about A_1, then A_3 falls on D_2 and D_3 on A_2. Also the points A_1, D_3, A_2, P are concyclic where P is the intersection of A_2D_2 and A_3D_3. Suppose that P lies inside $A_1A_2A_3$ then it easily follows that the three lines A_iD_i meet in a common point P such that the sides of $A_1A_2A_3$ subtend the angle of $120°$ at P which is called an isogonic center of the triangle. If P does not fall inside, the angles subtended at P are $60°$ or $120°$, see Johnson's *Modern Geometry*, p. 218. Denote by S_a and S_b the areas of $A_1A_2A_3$ and $B_1B_2B_3$ and let the origin of rectangular coordinates be A_1 with the x-axis along $A_1A_2 = a_3 > 0$, the order of rotation of A_1, A_2, A_3 being taken as positive. The coordinates $a_3, 0$; $X_3, Y_3 > 0$ of A_2, A_3 when used with the equation (1) and the expression for the area of $B_1B_2B_3$ give

$$(2\sqrt{3}\, X_3 + 10Y_3)a_3 - 2\sqrt{3}\,(X_3^2 + Y_3^2 + a_3^2) = 32S_b, \quad \text{or} \tag{2}$$

$$S_b = \frac{5}{8} S_a - \frac{\sqrt{3}}{32} \sum a_i^2. \tag{3}$$

Thus for given points A_1 and A_2 the locus of A_3 for which the B_1, B_2, B_3 are collinear is a circle which is easily constructed, and the line of collinearity goes through the centroid of $A_1A_2A_3$. The equation (2) says that the power of A_3 with respect to the circle, for which $S_b = 0$, is equal to $-16S_b/\sqrt{3}$. If A_3 is outside this circle then $S_b < 0$ and the order of rotation of B_1, B_2, B_3 is opposite to that of A_1, A_2, A_3 which has been taken as positive. If A_3 is inside the circle then $S_b > 0$ and the vertices of the two triangles have the same order of rotation. In this case there is a maximum S_b when A_3 is at the center $(a_3/2, 5a_3/2\sqrt{3})$, this maximum area is $\sqrt{3}a_3^2/12$ and the A triangle is isosceles.

We also find in a similar manner that

$$(A_1D_1)^2 = 2\sqrt{3}\, S_a + \sum a_i^2/2, \qquad \tfrac{3}{4}(A_1D_1)^2 = \sum a_i^2 - \sum b_i^2.$$

It is easily seen how to modify these results for the case where the equilateral triangles are drawn interiorly on the sides of $B_1B_2B_3$, and this gives the second isogonic center of the A triangle.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Michigan State College held an eight day school on Quality Control by Statistical Methods from June 9 to June 17, 1944, under the jurisdiction of the E.S.M.W.T. and the War Production Board.

Professor G. D. Birkhoff was awarded the honorary degree of Doctor of Science by Illinois Institute of Technology on June 18, 1944.

The following changes at Michigan State College are announced:

Assistant Professor J. M. Barbour has been promoted to an associate professorship.

Associate Professor W. D. Baten has been promoted to a professorship.

Dr. B. H. Bissinger has been granted a leave of absence to do mathematical work for the Army Air Forces.

Assistant Professor Paul Dressel has been appointed Chairman of the Board of Examiners and Director of Counselling.

Dr. W. L. Mitchell has been promoted to an assistant professorship and has been granted leave of absence to accept a position in the Aeronautical Research Department of the Bell Aircraft Corporation in Niagara Falls, N. Y.

Assistant Professor E. A. Nordhaus has been appointed major engineer in the Aerodynamics Unit of the Boeing Aircraft Corporation in Seattle, Washington.

Associate Professor J. E. Powell has been promoted to a professorship.

Assistant Professor C. P. Wells has been granted a leave of absence to do mathematical work for the Army Air Forces.

Assistant Professor E. T. Welmers has been granted a leave of absence to serve in the Flight Research Division of the Bell Aircraft Corporation in Niagara Falls, N. Y.

Assistant Professor J. W. Zimmer has been granted leave of absence to serve as senior development engineer with the Goodyear Aircraft Corporation in Akron, Ohio.

Assistant Professor G. E. Albert of Ohio State University has been granted a leave of absence to serve as research mathematician with the Lukas-Harold Corporation at Indianapolis, Ind.

Dr. P. H. Anderson of the War Production Board in Cleveland is now associated with the Bureau of Foreign and Domestic Commerce in Washington, D. C.

Associate Professor L. M. Blumenthal of the University of Missouri has been promoted to a professorship.

B. K. Brown of Colorado School of Mines has been granted a leave of absence to serve as a process instructor for the Tennessee Eastman Corporation.

Professor R. V. Churchill of the University of Michigan has been granted a leave of absence to serve as research specialist in the Army Air Force Instructors' School at Laredo, Texas.

Dr. Nancy Cole of Sweet Briar College has been appointed to an assistant professorship at Connecticut College.

Professor W. L. Duren, Jr. of Tulane University has been granted a leave of absence to serve as operations analyst for the United States Army.

Dr. J. E. Eaton of Queens College has been granted a leave of absence for work at the Radiation Laboratory at Massachusetts Institute of Technology.

Associate Professor W. W. Flexner of Cornell University has been granted a leave of absence to serve in the London office of the United Nations Relief and Rehabilitation Administration.

Professor C. W. Foard of Youngstown College has been appointed physicist at the Eastman Kodak Company in Rochester, New York.

E. L. Godfrey of Fenn College in Cleveland has been appointed to an assistant professorship at Northwest Missouri State Teachers College in Maryville, Missouri.

Dr. Theodore Hailperin of Cornell University has been appointed mathematician at Aberdeen Proving Ground.

Assistant Professor P. C. Hammer of Oregon State College has been granted leave of absence to serve with the Lockheed Aircraft Corporation of Burbank, California.

Dr. R. G. Helsel of Ohio State University has been granted leave of absence for war research at the Applied Physics Laboratory of the Johns Hopkins University.

Assistant Professor A. D. Hestenes of Carnegie Institute of Technology has been granted leave of absence to serve as research mathematician at the U. S. Naval Air Station in Patuxent River, Maryland.

Associate Professor M. R. Hestenes of the University of Chicago has been granted leave of absence to engage in war research.

Professor A. J. Hoare of the University of Wichita has retired.

Dr. Wilfred Kaplan of the University of Michigan has been granted leave of absence to serve as a research associate at Brown University, and has been promoted to an assistant professorship.

Dr. Irving Kaplansky of Harvard University has become a member of the applied mathematics group at Columbia University.

Dr. Mark Lotkin of Carleton College has been appointed to an assistant professorship at Wabash College, Crawfordsville, Indiana.

Assistant Professor Ingo Maddaus, Jr. of the University of Oregon is now serving in the Radiation Laboratory at Massachusetts Institute of Technology.

A. L. McCarty of San Francisco Junior College has retired.

Professor W. G. McGavock of Davidson College in North Carolina has been granted leave of absence to serve as visiting assistant professor at Duke University.

Assistant Professor Harriet F. Montague of the University of Buffalo has been promoted to an associate professorship.

Professor Richard Morris of Rutgers University has retired with the title of professor emeritus. Associate Professor E. P. Starke succeeds him as chairman of the mathematics department and has been promoted to a professorship.

Assistant Professor E. R. Ott of the University of Buffalo has been appointed executive engineer in the National Union Radio Corporation of Newark, New Jersey.

H. A. Palmer of the College of Idaho has been appointed Principal of the High School in Lund, Nevada.

Dr. A. M. Peiser of Cornell University has been appointed mathematician at Langley Field.

Professor W. G. Pollard of the University of Tennessee has been granted a leave of absence to serve at Columbia University with the N.D.R.C.

Assistant Professor H. A. Poppen of Illinois State Normal University has been appointed statistician with the firm of Stevenson, Jordan and Harrison of Chicago.

Dr. Louise Johnson Rosenbaum of Reed College has been promoted to an assistant professorship.

Associate Professor F. H. Steen of Allegheny College has been appointed Francis Asbury Arter Professor of Mathematics at that institution.

Assistant Professor N. E. Steenrod of the University of Michigan has been granted a leave of absence to work in the Office of Field Service in Washington, D. C.

Professor J. L. Synge of Ohio State University has been granted leave of

absence to serve as ballistics mathematician with the U. S. forces in the European theatre.

Professor T. Y. Thomas of the University of California at Los Angeles has been appointed professor of mathematics and chairman of the department at Indiana University. As chairman he succeeds Professor K. P. Williams.

Assistant Professor R. M. Thrall of the University of Michigan has been granted leave of absence to serve with the applied mathematics group at Columbia University.

Professor Agnes E. Wells of Indiana University has retired.

Dr. F. J. Weyl of Indiana University has been granted leave of absence to serve as mathematician in the Bureau of Ordnance of the Navy Department.

The following appointments to instructorships are announced:
Berea College, Kentucky: Mrs. Elizabeth C. Lukacs
College of the Holy Cross: Dr. V. O. McBrien
Emmanuel Missionary College (Berrien Springs, Michigan): C. L. Woods
Michigan State College: Elaine VanAken, Marian Michmerhuizen
The University of Notre Dame: Dr. Murray Mannos
The University of Pennsylvania: Dr. W. H. Gottschalk
The University of Rochester: W. N. Huff
The University of Virginia: W. R. Utz, Jr. (part-time)
Vassar College: Mrs. A. W. Goodman

Professor H. C. Hicks of Carnegie Institute of Technology died April 14, 1944.

Professor Emeritus David Eugene Smith of Columbia University died July 29, 1944. He was a charter member of the Mathematical Association.

Professor J. R. Wilton of the University of Adelaide, Australia, died during the week of April 20, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

UNIVERSITY OF CALIFORNIA ESMWT PROGRAM FOR INDUSTRY

Many favorable comments have been received in regard to the ESMWT program being conducted by the University of California in the Southern California area. The following report concerning the program has been written, upon invitation, by Professor W. M. Whyburn of the University of California at Los Angeles.

The heavy concentration of aircraft manufacture in Southern California and of shipbuilding in the San Francisco Bay and Los Angeles—Long Beach harbor areas has led naturally to extensive demands on the University of California for courses given under the Engineering, Science, Management War Training program of the U. S. Office of Education. More than one hundred thousand workers in vital war plants have benefited by these courses, and actual class enrollments at all times total near the ten thousand mark. For the Southern California area the University maintains major training centers on its Los Angeles campus, and in downtown Los Angeles, Burbank, Inglewood, Long Beach, and San Diego. A large number of additional courses are given in war plants and in such neighboring cities as Downey, Glendale, Huntington Park, Riverside, Santa Barbara, Torrence, Upland, and Wilmington.

Mathematics courses constitute a substantial part of the program offered in this region, and approximately sixty such courses are in progress at all times. Courses in mathematics are organized and supervised by Professors Clifford Bell and W. M. Whyburn. These supervisors maintain close personal contact with engineering and educational personnel of aircraft and other companies, and spend considerable time inside the plants finding training needs as they show up in the various departments of operation. They organize the courses, select the instructors, and are responsible for maintaining high instructional standards in each class. The teachers used in the courses are, for the most part, selected from the engineering personnel of the war industries although members of the Department of Mathematics of the University participate heavily in the instructional work, especially in the more advanced courses. In all cases, an effort is made to secure instructors with actual experience in the teaching of college mathematics. Experience has shown that instructors with college teaching experience are more successful in teaching these specialized war training courses than are men whose sole contact with mathematics has been their study of the subject together with their use of it in their daily work at the war plant. Students in the courses show a decided preference for experienced college teachers.

The course most needed by aircraft workers is one which has been designated as Engineering Mathematics I. This course reviews arithmetic, plane and solid geometry, elementary algebra, and then gives a reasonably complete treatment of plane and spherical trigonometry. Approximately ten thousand local workers have taken the course, and it has done much to give these people the working knowledge of the elementary mathematics that is a daily necessity in their jobs. This course is followed in regular sequence by Engineering Mathematics II, III, and IV; these cover the essential substance of analytical geometry, differential and integral calculus, and applications of these subjects. Several classes in Differential Equations I and II have been taught, and recently there has been a heavy demand for courses in trigonometry. In addition to these somewhat regular courses, there are several specialized courses that have been given for specific groups of employees. There is ample evidence that such courses have done much to speed up the production of better airplanes. A course "Aircraft Analytical

Geometry" has been a pioneering project and through it, lofting and layout work in aircraft and shipbuilding plants have been considerably revamped along more efficient and accurate lines. Many sections of this course have been offered in the major aircraft plants and at the various training centers. Recently, courses in statistical methods and quality control have been in heavy demand, and three or four different courses have been designed to meet these needs. For radio and electrical workers, courses ranging up to and including Heaviside Operational Calculus and Mathematics of Wave Propagation are being offered. Other advanced courses for engineering personnel include Calculus of Variations, Tensor Analysis, Mathematics for Stress Analysis, and Mathematical Aspects of Aerodynamics and Hydrodynamics.

NOTES ON THE NAVY V-12 PROGRAM

1. The following is an excerpt from a letter, dated May 11, 1944, addressed by Captain A. S. Adams to presidents of institutions participating in the Navy V-12 Program. More recent information indicates that the total decrease in the number of students in the V-12 Program upon November 1, 1944, will be about 28%.

"(The Bureau of Naval Personnel) plans to continue in effect during the V-12 term beginning 1 July 1944 the previously established overall unit quotas at all V-12 institutions. · · · Plans for the term beginning 1 November 1944, include a substantial decrease in the total number of students in the V-12 Program. A detailed statement concerning changes contemplated at your Unit will be sent you within the next month, in order that you may have as much advance notice as possible to plan the necessary readjustments in your instructional staff and the facilities now under contract to the Navy."

2. "Effective with the class which entered the Navy V-12 Program on 1 November 1943, all apprentice seamen, V-12(a), will receive a minimum of three terms of college training in the V-12 Program, regardless of the amount of college work completed prior to reporting for active duty at a Navy V-12 Unit. (The V-12(a) group which reported at Navy V-12 Units on 1 November 1943 will therefore complete college training on 1 November 1944, instead of on 1 July 1944, as originally scheduled.)

"A reduction in the future output of the pilot training program for the calendar years 1945 and 1946 has necessitated a rearrangement of the training schedule to meet the reduced output requirements; at the same time an opportunity is thus afforded to raise the standards of performance and improve the quality of pilot output. The rearrangement of the pilot training schedule is further affected by the closing of the Navy CAA-War Training Service and Naval Flight Preparatory Programs. Students withdrawn from the V-12 Program for flight training, on and after 1 November 1944, will enter the flight training program at Pre-Flight Schools, which constitute at present the third stage of the flight training program.

"Prior to April 1942, a minimum of two years of college credits was required for acceptance for flight training as aviation cadet, and the majority of aviation cadets were college graduates. By April 1942, because of the greatly accelerated pilot training program, it was found necessary to lower the educational qualification for aviation cadet to graduation from high school in order to meet the then scheduled pilot output. However, as the scheduled pilot output has now been reduced, it is considered desirable to bring the educational backgrounds for prospective pilots to substantially the same minimum level as was formerly required. It is considered in the best interests of the Naval Service that preparation to assume the responsibilities of a naval aviator be as thorough as practicable within the exigencies of the war. The additional semester of V-12(a) training is intended to enhance the military and educational background of prospective aviation cadets to the end that the duties they will be expected to perform as naval aviators may be carried out more effectively."—*Navy V-12 Bulletin No. 213 (Subject C)*.

3. It was announced in the MONTHLY, March, 1944, that enlisted men gathered at assembly points before assignment to V-12 institutions would be given some review work in mathematics. This Pre-V-12 Program was not inaugurated until May. During April, an orientation program for Pre-V-12 instructors was held. Complete details of this important project will appear in an early issue of the MONTHLY.

DEFERMENT OF MATHEMATICIANS AGES 26 THROUGH 37

A man of age 26 through 37 may ("shall" in the case of men ages 30 through 37) be retained or replaced in Selective Service Class II-A if he is found to be "necessary to and regularly engaged in" an activity in support of the national health, safety, or interest, or in Class II-B if he is found to be "necessary to and regularly engaged in" an activity in war production. (The term "necessary" is omitted in the case of men 30 through 37.)

In view of the fact that all Activity and Occupation Bulletins have been rescinded, the question has been raised whether professional mathematicians are now regarded as essential within the meaning of the previous paragraph. The following sections bearing upon the question have been taken from the "List of Essential Activities" published May 12, 1944, by Selective Service.

Educational Services.—Public and private industrial and agricultural vocational training; elementary, secondary, and preparatory schools; junior colleges, colleges, universities, and professional schools, educational and scientific research agencies; United States Maritime Service Training Program; Civil Aeronautics Administration Civilian Pilot Training Program; armed forces contract flying, ground, and factory aviation schools; and the production of technical and vocational training films.

Technical, Scientific, and Management Services.—The supplying of technical, scientific and management services to establishments engaged in war production; publication of technical and scientific books and journals.

CODE FOR STUDENTS ON FORM 42-A SPECIAL

The *Dictionary of Occupational Titles* used by the United States Employment Service does not provide a code for students to be used in filling out the line reading, "United States Employment Service Dictionary Code (Occupation)," on DSS Form 42-A (Special) of the Selective Service System. For this purpose, the following code for students of science has recently been prepared by the National Roster and approved by the National Headquarters of the Selective Service System.

Physical Sciences

0-40.34 Geology
0-40.37 Meteorology
0-40.38 Chemistry
0-40.43 Mathematics
0-40.44 Physics
0-40.46 Geophysics

Engineering

0-40.47 Civil
0-40.48 Agricultural
0-40.50 Ceramic
0-40.51 Marine and Naval Architecture
0-40.52 Sanitary and Public Health
0-40.54 Industrial and Administrative
0-40.55 Chemical
0-40.56 Electrical
0-40.57 Mechanical
0-40.60 Metallurgical
0-40.61 Aeronautical
0-40.62 Mining
0-40.63 Petroleum
0-40.64 Radio
0-40.65 General & Other

THE AST RESERVE PROGRAM

A total of 166,761 men took the "Army-Navy Qualifying Examination" on March 15, 1944. Of this number, approximately 63,000 were seventeen-year-olds within the age range specified for the Army and Navy College Reserve Programs. Only an estimated 21,000 of this latter number, however, expressed Army preference, and, on the basis of experience, it can be anticipated that another 20% will not pass the physical examination. As a result, the AST Reserve Program seems destined to be smaller than the Army and institutions under contract to the Army had anticipated.

Several devices are being employed by the Army to increase the number of students registered in the AST Reserve Program. For instance, a passing mark was established for the latest qualifying examination which was lower than that originally required for the ASTP. Also, a large number of men who passed the examination and who expressed Navy preference have been given an opportunity to volunteer for the ASTRP without reexamination; this was made possible by the fact that the Navy decided to draw a very small quota from this source for the July term. Another potential source of recruits for the program is the large number of seventeen-year-olds who previously volunteered for the Air Force; since the Army Air Force College Program was completely eliminated by July 1, such men are being urged to enter the ASTRP.

Even though every encouragement has been given to the development of the

Reserve Program, it appears that the total number of men registered will only be a small fraction of the total withdrawn from the colleges with the elimination of the AAF Program and the reduction of the ASTP. Consequently, many institutions previously under contract to the Army cannot be utilized in connection with the ASTRP. Institutions selected to continue have been determined on the basis of a plan of priority developed by the Army. The order of assignment of ASTRP Units to institutions who have had AST or AAF Programs is (1) military colleges, (2) ROTC colleges and universities, and (3) men's colleges.

The ASTRP will be observed with considerable interest by educators. The men are still civilians, and can withdraw from the curriculum at their own discretion. Moreover, the Army has made no attempt to classify the registrants on the basis of special abilities, so institutions must accept responsibility for educational counseling. The program of study has not been rigidly prescribed, and some courses are being offered which are below the usual level of college work. If the Program is successful, it is probable that it will be continued over a considerable period of time, and perhaps expanded upon a broad basis.

ACTIVITIES OF THE COMMITTEE ON WAR TRAINING PROGRAMS

WILLIAM L. HART, University of Minnesota

During the summer of 1943, the War Policy Committee of the A.M.S. and the M.A.A. appointed a Subcommittee on War Training Programs. The Subcommittee now consists of Professors C. R. Adams (Brown); Ralph Beatley (Harvard); H. M. Bacon (Stanford); B. H. Brown (Dartmouth); H. J. Ettlinger (Texas); W. L. Hart (Minnesota), Chairman; C. V. Newsom (Oberlin); W. M. Whyburn (California). This Subcommittee decided to record its activities in the form of a series of reports, each dealing with the mathematical part of some specified war program. By the time the Subcommittee commenced its work, the curricula of the service schools maintained by the Armed Forces were thoroughly established and, in many cases, the flow of trainees was already beginning to decrease rapidly. Hence such curricula were eliminated from consideration by the Subcommittee. It decided to restrict its attention to war programs of the Armed Forces involving mathematical instruction of approximately collegiate grade carried on under the control of college departments of mathematics.

The following programs have been considered and the Subcommittee has presented reports on them to the War Policy Committee.

1. Mathematics in the College Training Program (Aircrew) of the Training Command of the Army Air Forces (report drafted by Professor Hart, November, 1943).*

2. Mathematics in the Army Specialized Training Program (report drafted by Professor Bacon, February, 1944).*

* The chairman of any collegiate department of mathematics can obtain a mimeographed copy of the report by writing to the office of Professor J. R. Kline, American Mathematical Society, University of Pennsylvania.

3. Mathematics in the College Training Program, Navy V-12, of the Training Division, Bureau of Naval Personnel (report drafted by Professor Newsom, March, 1944).*

When the Subcommittee commenced active operations, the Pre-meteorological Programs of the A.A.F. were in their final stages. Hence, in lieu of a report, the Subcommittee made arrangements with the Academic Office of the Programs to provide the War Policy Committee with copies of the very complete descriptive booklet† on the Programs which was issued by the University Meteorological Committee.

The War Policy Committee, through the office of Professor Kline, sent mimeographed copies of Reports 2 and 3 and the information booklet on the Pre-meteorological Programs to the officers of the A.M.S. and M.A.A. and to a selected list of administrators of departments of mathematics. Copies of Reports 2 and 3 also were sent to appropriate offices of the Armed Forces.

At the present time, the Subcommittee is giving consideration to the mathematical part of the program of the United States Armed Forces Institute. As a result of activity of the Subcommittee, Dr. Ralph W. Tyler, the Director of the Examinations Staff for U.S.A.F.I., cordially invited mathematicians to appoint a committee to give advice on mathematical problems in connection with examinations to be sponsored by the U.S.A.F.I. In accordance with this invitation, in June, 1944, Professor M. H. Stone, Chairman of the War Policy Committee, appointed a Subcommittee on Examinations to advise the staff of U. S.A.F.I. This Subcommittee has commenced active operations; its membership is as follows: Professors Ralph Beatley (Harvard); L. L. Dines (Carnegie Institute of Technology); W. L. Hart (Minnesota); C. C. MacDuffee (Wisconsin); W. T. Reid (Chicago), Chairman.

The Subcommittee on War Programs, through its chairman, will welcome at any time comments concerning mathematics in the various war programs.

REPORT ON MATHEMATICS IN THE V-12 PROGRAM

This report, to which reference was made in the previous article by Professor W. L. Hart, has been approved by the War Policy Committee. The report was written on the basis of the experience of the participating colleges up to March 1, 1944.

1. *General description of the Navy V-12 Program.* The Navy College Training Program (V-12) was inaugurated upon July 1, 1943. Its purpose, as phrased by the Navy Department, is to produce Naval officers chosen from high school seniors, high school graduates, and college students who appear to have the potentialities for ultimate selection as officers after carefully prescribed college

* Published as the next article in this MONTHLY.

† The Academic Office of the Programs is now closed. It sent copies of the booklet and amplified outlines of the pre-meteorological courses to the offices of admissions of colleges throughout the United States. Hence, the booklet and outlines should be accessible to every interested mathematician. At present, a few copies of the booklet remain in the hands of Professor Kline.

training. This college training shall be carried on while the men are on active duty, in uniform, receiving pay, and under general military discipline.

The Navy has insisted that all colleges and universities holding Navy V-12 contracts for undergraduate instruction offer three sixteen-week terms in each calendar year. In fact, there must be sixteen full weeks of instruction in each term, exclusive of vacations and holidays.

The several types of officer candidates to receive training in the V-12 Program are as follows: aviation candidates, deck candidates, supply corps candidates, pre-medical and pre-dental corps candidates, general engineering candidates, engineering specialist candidates, pre-chaplain corps candidates, and aerology specialist candidates. Prescribed curricula, which vary in length from two to eight terms, have been outlined for these various programs of training; for information upon the curricula, reference may be made to Navy V-12 Bulletin No. 101. All curricula contain at least two terms of elementary mathematical analysis. Moreover, technical programs involve two terms of calculus, and, in some cases, there is the equivalent of a term of differential equations.

Topical outlines for the individual courses composing the curricula have also been provided; they were drawn up with the assistance of academic consultants. The original outlines for the courses in mathematics were published in this MONTHLY, December, 1943. Since that date, no essential change has been made in their content except for the courses in navigation and nautical astronomy.

The curricula and courses, as prescribed, are for students who enter the V-12 Program as freshmen, without previous college work. Students entering the Program with advanced standing, based on previous college work, are not required to follow the prescribed curricula. Such transfer students are urged, however, to take as many of the prescribed subjects as possible, and certain minimum requirements for V-1 and V-7 transfers must be met. As a result of the liberal demands made of transfers, few institutions have had actual experience as yet with many prescribed V-12 courses above the freshman level.

2. *Selection and qualifications of the V-12 trainees.*

(a) *Transfers from other Navy College Programs.* Upon July 1, 1943, approximately 80% of the V-12 trainees assigned to institutions were students already in college; these students were previously enlisted in class V-1 or V-7, or held probationary commissions in the U. S. Naval Reserve or were enlisted in the Marine Corps or Coast Guard Reserves. Trainees in this general classification were a heterogeneous lot with considerable range of ability, since no rigorous academic criterion was a part of the basis for their selection.

(b) *Transfers from V-5.* V-5 candidates are selected by Naval Aviation Cadet Selection Boards. When temporarily transferred to V-12, as an incident in the adjustment of aviation training quotas, such trainees are allowed to take only the first two terms. After the completion of two terms, the qualifications of this group of men are subject to review, and, if found still qualified, the men are reclassified as V-5, and they then proceed to aviation training. Many V-12 trainees in this category appear to be misfits in the Program inasmuch as Naval

Aviation Cadet Selection Boards are admitting students for training who have had very little previous work in mathematics and science.*

(c) *Freshmen who qualify through the Army-Navy tests.* High school seniors in their last semester and high school graduates, within the age classification of seventeen to twenty-two years, may become candidates for the Army A-12 or for the Navy V-12 by successfully passing one of the "Army-Navy tests." Students taking the tests may choose the branch of service which they prefer. Approximately 300,000 young men took the first test given April 2, 1943, and about 78,000 took the second one on November 9, 1943; other tests will be given periodically in the future. Persons familiar with the first test generally believe that it was an adequate device for the elimination of young men of low intelligence. Unfortunately, there was insufficient emphasis upon mathematics, and many V-12 trainees, who successfully passed the first tests, are now having great difficulty with the mathematics in the Program. The second test was weighted more heavily in mathematics, and students in this category, who entered the V-12 Program in March, should be better prepared.

(d) *Men who have previously been on active duty.* The Navy has determined upon a policy of selecting some men on active duty for the V-12 Program. Present indications are that the quota in this category is to be considerably increased. Until recently, the only academic qualification required of such men was high school graduation; now, each man selected must have had a minimum of two years of high school mathematics, and must have made a high score upon a general classification test. In addition, it is understood that steps are now under way to give this group of trainees some refresher training in basic academic subjects before their enrollment in V-12.

Some V-12 trainees first selected in this category had no previous training in algebra or geometry, and a large number had less than two years of secondary mathematics. For such men, the courses in mathematics and science in the V-12 Program are obviously too advanced.

3. *Academic load of trainees.* All students in the V-12 Program are expected to be enrolled for a minimum of seventeen hours of academic work each term. Curricula 101 and 201, followed by most trainees during the first two terms, involve two laboratory subjects, namely, physics and engineering drawing, so the total number of contact hours per week in class and laboratory is twenty-three. This latter total increases to thirty, or more, in the last terms of some of the technical curricula, but the total of contact hours plus study hours should remain approximately the same.

The Navy has assumed that each lecture-recitation period would require approximately two additional hours of study and that the laboratory periods

* Since writing this report, Navy V-12 Bulletin No. 213 (Subject C) has appeared. This Bulletin specifies that Naval aviation students classified in V-12 will now receive a minimum of three terms of college work. Such a step may tend to aggravate present problems resulting from the deficiencies in mathematics and science to be found in this group of men, especially if they are to be classified as Deck Candidates.

would not require outside preparation. Thus, on the average, it is expected that a student will regularly devote about fifty-five hours per week to academic work.

4. *Class work of trainees.* In addition to the academic load, each V-12 trainee is required to devote about ten hours a week to physical training. Moreover, in spite of the utmost cooperation on the part of Naval administrative personnel, the term "military discipline" implies an additional demand upon the trainee's time. As a consequence, the total Program is definitely heavy, and many students have had great difficulty adjusting themselves to a situation which requires much more concentrated effort than other experiences to which they have previously been accustomed. When questioned, instructors generally have admitted that lessons assigned for preparation outside of the classroom must be somewhat shorter than usual; otherwise, the work is not completed. On the whole, however, it appears that students *with proper background* are able to handle the Program in a satisfactory manner, and there is no reason for any extensive modification of its content. The attitude and interest of trainees in the class room is generally reported to be good.

5. *Separation of trainees from Program.* Navy V-12 Bulletin No. 101 states, "Cases warranting separation from the program for scholastic reasons will be referred to the commanding officer for action." The experience of institutions in this connection has been very good. Upon the recommendation of scholastic officers of an institution, the Naval officer in command has generally taken immediate steps to remove the student in question from the Program. The Navy has used wisdom in awarding extensive authority to commanding officers in anticipation of the necessity for separating obvious misfits from the V-12 Program; this is in decided contrast to the policy in some of the other government-sponsored educational programs.

6. *Academic features of the Program.* The Navy has insisted that the V-12 Program is essentially a college program, and the usual standards of the contracting institution are to be maintained. No instance is known of any interference on the part of Navy authorities with the prerogatives of civilian teaching personnel. It is specified in Navy Bulletins that an institution will set examinations according to its own practices; the instructor in each course will select the text; and so on.

As already indicated, the Navy does specify the minimum content of courses through the medium of topical outlines supplied to contracting institutions. Repeated warnings have been issued that any fundamental deviation from prescribed patterns represents a serious threat to the program as a whole in view of the fact that there must be some transfer of students from one institution to another, especially at the end of the first two terms. It is possible, however, to rearrange the sequence of course content.

In addition to examinations given under institutional auspices, the Navy gives "a qualifying examination of its own to all first college year students during their second term, and may give other examinations at prescribed intervals for purposes of selection." The scores made on these tests are used to supplement

information otherwise available before assigning trainees to designated curricula at the end of the second term.

The first of these qualifying tests was given during the latter part of November, soon after the start of the second term. Instructors generally did not have an opportunity to look at the examination. Two reputable mathematicians, who did inspect the test after it was given, write, "(We) both felt that this examination was quite suitable and satisfactory for the purposes for which it was presumably intended." The part devoted to mathematics seems to have been constructed to provide a rough estimate of a trainee's knowledge of trigonometry.

The percentile scores of individual trainees, calculated upon a national basis, have been provided the various participating institutions for the one national examination so far given. Up to the present time, however, the Navy has issued no publicity in regard to the relative standings of the various college V-12 detachments. Moreover, to the best of our knowledge, such information has not been made available confidentially to academic authorities of the colleges or to commanding officers of the V-12 detachments. The Subcommittee does not know to what use, if any, the Bureau of Naval Personnel will put the available statistics about the examination scores, outside of employing them as a partial basis for assigning trainees to later duties.

7. *Problems relative to trainees who have inadequate preparation in mathematics.* The work in mathematics for the first two terms of the V-12 Program has been organized in two optional sequences of courses, namely, M1–M2, and M3–M4. Courses M1 and M2 are somewhat more elementary than M3 and M4, and the outlines of M3 and M4 include topics not suggested for M1 and M2. By Navy instruction, the M1–M2 sequence "is designed for students who enter with two or less units of mathematics or who, regardless of the number of units of previous preparation, are, in the judgment of the college authorities, not adequately prepared to undertake Mathematical Analysis 3, 4."

The situation pertaining to the beginning courses in mathematics has been considerably complicated by the admission of a large number of students into the Program who could not even cope with M1 and M2. This fact was acknowledged in Navy Bulletin No. 66, issued during the summer of 1943. Quotations from this bulletin follow: "It appears that some students have entered the V-12 Program whose preparation in mathematics is inadequate even for Mathematics 1. Colleges and universities are hereby authorized to establish the necessary refresher or makeup classes for such students, and to make necessary readjustments in the content and hours of subsequent classes in mathematics during the first two terms. It is required, however, that such students complete the substantial equivalent of Mathematics 2 by the end of their second term." "It is not contemplated that any V-12 student shall be permitted to remain in college more than the prescribed number of terms because of deficiency in high school preparation. He must, with necessary refresher classes and other adjustments in the curriculum—make up his deficiencies and complete all prescribed courses by the end of his second term in college, or be subject to separation from the program."

To ask a student to make up serious deficiencies in mathematics during the same short period of time that he must complete the content of an extremely heavy program of prescribed courses represents virtually an impossible request. Available data indicate that instructors have been able to salvage only a small fraction of such students, even after a tremendous amount of "sweat and tears" on the part of both students and faculty. Some trainees who were previously on active duty have been especially resentful of the situation; for instance, one said, "I was placed in the V-12 Program almost against my wishes; the assignment given me in the Program is impossible for me to handle, so now I shall have a permanent black mark on my record."

In the case of students who are finally certified at the end of two terms as having successfully passed the equivalent of M1 and M2, it is problematical whether many of them should continue to the calculus, at least, on the level of M5 and M6 as presently organized. Also, some students completing courses M3 and M4 have shown so little mathematical aptitude that it is doubtful that they should be required to study a standard course in calculus. It is apparent, therefore, that reconsideration should be given to the mathematics requirement for the third and fourth terms in certain technical curricula. The need for possible revision is especially urgent in the curricula for Deck Candidates and in the N R.O.T.C. general curricula. It may be assumed that there is no need for such revision in the engineering curricula, because it has been specified in the Navy Bulletins that "all engineering candidates shall be expected to be qualified for and to complete satisfactorily Mathematical Analysis III and IV!"

If the Navy expects to continue the present policy of requiring the calculus during the third and fourth terms for *all* trainees who are Deck Candidates or are students in the N.R.O.T.C. (general), and who have completed M1–M2 or M3–M4 as now organized, it would be desirable to consider the creation of a new course involving the calculus for those trainees who have demonstrated limited understanding or ability in mathematics. This provision might involve giving mathematical instruction at two levels for the first four instead of just the first two terms in some institutions.

8. *Comments on the organization of mathematics courses.* The outlines provided for the various courses in mathematics have now been subjected to the "test of experience." Although it is evident that the courses were organized by persons having considerable understanding of the entire situation, defects have since been observed by competent teachers and administrators, and the course outlines are being criticized on various grounds.

Probably most mathematicians prefer a program which is not too rigid, but this does not imply that the course outlines necessarily should be extremely brief and indefinite. In the hands of a conscientious teacher, an outline which clearly specifies minimum content may be a more flexible instrument than a briefer but indefinite description of the material to be covered. Moreover, in a Program where there is considerable shifting of students between terms from one institution to another, all outlines should be sufficiently specific in regard to

minimum content that it is possible for instructors to make plans for such continuation courses as M4 and M6. Experience has revealed that the course outlines can also be improved with respect to choice of material. In this regard, there appears to be considerable agreement among mathematicians relative to possible deletions and additions. It is undoubtedly true that mathematical consultants familar with V-12 and the experience of mathematics teachers participating in the Program could effectively improve the outlines of the mathematics courses.

Since M1 and M2 are for students with limited background, it is essential that every opportunity be given such students to become familiar with basic algebraic concepts. Instructors comment that they do not have ample time for necessary drill. It appears that the elimination of some topics from the outlines could be accomplished without destroying their effectiveness; then more time could be devoted to a thorough study of elementary notions. A student with only two units of high school mathematics is still very immature, mathematically. Consideration might be given to the elimination of the introduction to spherical trigonometry from the M2 outline, since it is presumed that trainees in this course will continue to non-technical curricula or to the curricula for deck candidates; in the latter case, a term course in spherical trigonometry is provided. Also it appears that the treatment of such topics as simultaneous quadratics, theory of equations, and complex numbers, could be quite brief.

It is noteworthy, also, that some important topics have been omitted from the M1–M2 outlines. In particular, it is unfortunate that provision has not been made for a treatment of the binomial theorem and the geometric progression. Moreover, some study of inequalities would be useful. If students completing M2 are to continue to the calculus, an early introduction to tangents and normals would be desirable.

Some minor adjustments might well be made in the M3–M4 outlines. It is the opinion of some mathematicians, for instance, that the time spent on permutations, combinations, and probability might be devoted to a consideration of inequalities and partial fractions.

Limited experience with courses M5–M6 indicates that a satisfactory distribution of the topics in the calculus has not been worked out between the two terms. It is virtually impossible to cover the work of M5 in a four-hour course with any semblance of thoroughness, whereas there is more than enough time to do rather satisfactory work in M6. Some of the difficulty may arise from the vagueness present in the outlines, inasmuch as it is not clear how detailed the treatment of the indefinite integral should be in M5.

It is somewhat surprising that approximate methods are ignored in connection with integration. This appears to be the most critical omission for a group of students who are seriously concerned with applications.

A very awkward situation exists in connection with courses M5 and M6 in view of the fact that the work of each term has been outlined upon the assumption that four class hours a week are to be devoted to the work. Instead of the

4–4 distribution, however, some curricula follow a 5–3 plan, and others have only the equivalent of a 4–3 distribution. In other words, some institutions are teaching the calculus simultaneously upon a 4–4, a 5–3, and a 4–3 plan. As a result, unnecessary complications are arising in the keeping of records, in the economical utilization of staff, and in the organization of teaching programs. It appears that only slight adjustments are necessary in the various curricula to permit the introduction of a uniform teaching program for courses M5–M6.

The outline of M7 sketches in general terms a typical first course in differential equations, based on elementary calculus. The great flexibility present in the outline appears to be consistent with the apparent purpose of the course.

Courses M8 and M9 in elementary navigation and nautical astronomy are listed in the mathematics section of Navy Bulletins, and presumably are handled in most institutions by mathematics departments.

The descriptions of these two courses provided to instructors previous to July 1, 1943, indicated that the Navy desired a technical presentation of navigation and nautical astronomy. Many institutions made serious efforts to obtain instruments as well as the elaborate sets of tables mentioned in the outlines. In fact, without some equipment, the courses as outlined could not be presented.

A few weeks after the start of the first term, Navy Bulletin No. 48 appeared. The following quotation is from that Bulletin:

"Navigation and Nautical Astronomy I and II are to be basic courses and are not designed to give any technical skill in the science of navigation. Practical navigation training is to be reserved for Naval Reserve Officer Training Corps and Midshipmen Schools. The purpose of Navigation I is to give the basic mathematics with emphasis on the elements of spherical trigonometry and elementary vector mathematics necessary for the student to comprehend the navigation courses he will be given later under Navy instruction. Problems may be drawn from nautical situations to keep the work alive but the emphasis shall be upon mathematical principles only. Any elementary text in Spherical Trigonometry would be an adequate guide to what is desired. The purpose of Navigation II is to give the student the astronomical background necessary to an understanding of the principles of celestial navigation and inherent in the practice of celestial navigation. In essence this course shall be a course in elementary nautical astronomy. Any elementary text in nautical astronomy would be an adequate guide to what is desired. Since Navigation and Nautical Astronomy I or II are not courses in practical navigation, equipment as charts, navigational instruments, and other devices for the study of practical navigation will not be necessary."

The outlines of M8 and M9 provided in Navy V-12 Bulletin No. 101, issued November 1, 1943, express the same point of view as that set out in Bulletin No. 48. It is unfortunate that a more thorough analysis of the needs and purposes of these two courses could not have been made before the inauguration of the Program. In many institutions, a change of text books was necessary before the start of the second term (November 1, 1943), and other adjustments had to be made.

Mathematicians generally are not familiar with the methods employed by the Navy in the field of celestial navigation. Hence, it would be desirable to have expanded outlines for the material which the Navy wishes taught in M8 and M9.

Summary of Recommendations

A. It is recommended that the Training Division, Bureau of Naval Personnel, obtain expert mathematical assistance to improve the course outlines for mathematics in the V-12 Program in the light of a careful analysis of the opinions of teachers in the participating colleges. This recommendation is not to be construed as a suggestion that the outlines should be revised in such a manner as to eliminate their present flexibility, which is a commendable feature.

B. In the case of trainees who come to the Program from active duty in the Navy, it is recommended that the Division give these men suitable refresher training, whenever a considerable interval of time has elapsed since their last academic studies or whenever mathematical tests show this to be desirable, before they commence any part of the regular curricula of the Program.

C. The Training Division is advised that problems of a serious nature are resulting from the plan followed at present in certain curricula of attempting to bring together in M5 two groups of students with distinctly different mathematical backgrounds. The members of one group enter the V-12 Program with adequate mathematical preparation, and complete courses M3 and M4 previous to registration in M5. Students in the other group enter the Program with deficiencies in mathematical preparation, and take M1 and M2 before M5. It seems unrealistic to expect the average member of the latter group to be able to carry successfully the work of M5 if it is taught at a level appropriate for most of the students who have had M3 and M4. In general, then, it would seem to be necessary either to increase the total time spent on mathematics by the members of the poorer group or to diminish their ultimate goal. Hence, we recommend:

C_1. That renewed consideration be given to the curriculum during the first two terms for men who must take M1–M2, and to the placement of these men in curricula for the third and fourth terms.

C_2. That curricula involving four terms of mathematics be re-examined in the light of the fact that many students who have passed M1–M2 (or M3–M4) do not give evidence of mathematical maturity sufficient to profit by the study of an extensive course in the calculus. It is suggested that especial attention be given to the curricula for Deck Candidates and for the General R.O.T.C.

D. If at any time the Training Division should decide to set comprehensive national examinations in specific courses in mathematics in the Program, it is recommended that the Division should consult competent mathematicians in the preparation of the tests. Such men should possess wide experience as teachers of mathematics at the undergraduate level and should have keen interest in examination techniques.

E. It is urged that consideration be given to the slight reorganization required of some technical curricula to permit the teaching of M5 and M6 upon a uniform time schedule for all curricula.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following thirty-four persons and one institution have been elected to membership on applications duly certified:

To Institutional Membership

McGill University, Montreal, P. Q., Canada

To Individual Membership

L. E. BABCOCK, M.S. (Illinois) Asso. Prof., Newberry Coll., Newberry, S. C.

R. A. BRADLEY. Student, Queen's Univ., Kingston, Ont., Can.

A. T. BRAUER, Ph.D. (Berlin) Asso. Prof., Univ. of North Carolina, Chapel Hill, N. C.

N. E. DODSON, A.B. (Berea Coll.) Instr., Newberry Coll., Newberry, S. C.

BROTHER DOMINIC, A.B. (St. Mary's Coll., Calif.) Instr., St. Mary's Coll., St. Mary's College, Calif.

W. S. ERICKSON, Ph.D. (Wisconsin) Prof., State Teachers Coll., Minot, N. D.

ELVY L. FREDRICKSON, A.B. (Willamette) Grad. Asst., Univ. of Colorado, Boulder, Colo.

R. A. GOOD, A.M. (Wisconsin) Acting Instr., Univ. of Wisconsin, Madison, Wis.

W. H. GOTTSCHALK, Ph.D. (Virginia) Asst. Prof., Univ. of Richmond, Richmond, Va.

LAURA Z. GREENE, M.S. (Chicago) Instr., Washburn Univ., Topeka, Kans.

J. R. HAMMOND, A.M. (Harvard) Asst. Prof., U. S. Naval Acad., Annapolis, Md.

M. A. HANHAUSER, A.B. (St. Bonaventure) Instr., St. Bonaventure Coll., St. Bonaventure, N. Y.

MRS. AUGHTUM S. HOWARD, Ph.D. (Kentucky) Asso. Prof., Kentucky Wesleyan Coll., Winchester, Ky.

M. A. HYMAN. Student, Univ. of Maryland, College Park, Md.

SAMUEL KARLIN, B.S. (Ill. Inst. of Tech.) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill.

JULIUS KLEWANSKY, A.B. (Brooklyn) Recently Instr., Univ. of New Hampshire, Durham, N. H.

R. P. LALONDE. Inspector, R.C.A. Victor Co., Montreal, P. Q., Can.

ELLA E. LAUSMAN, A.M. (Michigan) Instr., Western Michigan Coll., Kalamazoo, Mich.

L. C. LAY, A.M. (Southern California) Design Engr., Lockheed Aircraft Corp., Burbank, Calif.

MRS. MARGARET B. LEHMAN, A.M. (U.C.L.A.) Instr., Univ. of California at Los Angeles, Los Angeles, Calif.

EMANUEL LEVINE, A.M. (Columbia), Ed.M. (Rutgers) Teacher, Gen. Sci., High School, Freeport, N. Y.

MARK LOTKIN, Ph.D. (Kiel, Germany) Instr., Carleton Coll., Northfield, Minn.

M. H. MARTIN, Ph.D. (Johns Hopkins) Prof., Chm. of Dept., Univ. of Maryland, College Park, Md.

K. A. MCMURTRIE, B.S. (Brown) Cpl., Officer candidate, U. S. Army Signal Corps.

R. S. NICHOLLS, B.Sc. (London) Chemist, Miles Laboratories, Inc., Elkhart, Ind.

W. J. ROBINSON, Ph.D. (Ohio State) Asst. Prof., Allegheny Coll., Meadville, Pa.

L. D. RODABAUGH, Ph.D. (Ohio State) Visiting Lecturer, Oberlin Coll., Oberlin, Ohio

BROTHER ROLAND, B.S. (Laval) Prof., Secretary, Superior School of Commerce, Quebec, P. Q., Can.

HANS SAMELSON, Dr.sc.math. (Fed. Inst. of Tech., Zurich) Asst. Prof., Syracuse Univ., Syracuse, N. Y.

M. ANICE SEYBOLD, A.M. (Illinois) Instr., Illinois Coll., Jacksonville, Ill.

E. B. STRUTTON, M.S. (Toledo) Instr., Univ. of California at Los Angeles, Los Angeles, Calif.

MARY M. TEMPLETON, A.M. (North Carolina) Prof., Lander Coll., Greenwood, S. C.

J. H. WHITE, A.B. (Princeton) Recently Instr., Lafayette Coll., Easton, Pa.

W. A. WILLIS, A.B. (Johns Hopkins) Mech. Engr., Bendix Aviation Corp., Towson, Md.

THE SPRING MEETING OF THE IOWA SECTION

The thirty-second meeting of the Iowa Section of the Mathematical Association of America was held at the Montrose Hotel in Cedar Rapids, Iowa, on the afternoon of Saturday, April 15, 1944, in conjunction with the meeting of the Iowa Academy of Science. Professor N. B. Conkwright, Chairman of the Section, presided.

The attendance was about thirty, including the following twenty members of the Association: E. W. Anderson, J. W. Beach, J. H. Butchart, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, W. M. Davis, Cornelius Gouwens, Gertrude A. Herr, J. F. Heyda, Dora E. Kearney, A. T. Lonseth, R. B. McClenon, F. M. McGaw, E. N. Oberg, E. R. Smith, L. W. Swanson, Henry Van Engen, L. E. Ward, Roscoe Woods.

In the forenoon members of the Section attended the meeting of the Iowa Academy of Science, at which Professor E. R. Smith delivered a paper entitled *Mathematical continuations corresponding to physical impossibilities* as his Academy presidential address.

At the business meeting the following officers were elected for the next year: Chairman, J. H. Butchart, Grinnell College; Vice-Chairman, E. N. Oberg, University of Iowa; Secretary, Cornelius Gouwens, Iowa State College.

The following papers were presented:

1. *The mathematics situation in the high schools before Pearl Harbor*, by Professor Henry Van Engen, Iowa State Teachers College.

2. *The mechanism of gravitational force*, by Professor E. E. Watson, Iowa State Teachers College, introduced by the Secretary.

This paper will be published in the *Proceedings of the Iowa Academy of Science* for 1944.

3. *A design for sampling Iowa families*, by Mrs. Bernice Brown, Iowa State College, introduced by the Secretary.

This speaker gave a description of the sampling design devised for the Iowa canning and gardening survey. The paper included an explanation of method of selecting the sample, and a partial evaluation of the technique followed. This paper will be published in the *Proceedings of the Iowa Academy of Science* for 1944.

4. *A simple treatment of perturbations in linear algebraic systems*, by Dr. A. T. Lonseth, Iowa State College.

The speaker gave a brief derivation of the principal inequality (10, p. 335) in his paper on *Systems of linear equations with coefficients subject to error*, which appeared in the Annals of Mathematical Statistics, vol. 13, 1942, pp. 332–337.

5. *Repetitious numbers*, by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

Professor Allen defined a repetitious number as one which, written decimally, has at some point every possible finite succession of digits. He pointed out that

every such number has then each such succession an infinite number of times. An example of a repetitious number was given, and methods for obtaining others were outlined. For instance, he explained a method for obtaining from a single one a set which can be put into a one-to-one correspondence with all real numbers.

6. *A mechanical calculator to solve the equation* $x=(d+z+c)y/(b+y)-c$, by Dr. Zabog V. Harvalik, Duluth State Teachers College, Duluth, Minnesota, introduced by the Secretary.

In this paper it was pointed out that the above equation can be written in the form $(x+c)/y=a-(x+c)/b$, in which form it can be interpreted geometrically by consideration of two similar triangles with a common vertex. The mechanical calculator constructed for the purpose of solving the equation was based upon this idea. The calculator was primarily designed to locate foreign bodies by a parallactic method employed in connection with X-rays. This paper will be published in the *Proceedings of the Iowa Academy of Science* for 1944.

CORNELIUS GOUWENS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Annual Meeting, Chicago, Ill., November 24–26, 1944.

The following is a list of the Sections of the Association with dates of future meeting so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

A Complete, Self-Contained Course

MARINE AND AIR NAVIGATION

Stewart-Pierce

Treats clearly, interestingly, every phase of navigation. Includes *more* navigation—requires *less* background—than other texts. Identifies navigators' stars quicker; clearer, more logical treatment of essential relative-motion problems. Has many tables and charts not usually available—including new star charts based on new Transverse Mercator Projection. $4.50

ANALYTIC GEOMETRY

Steen-Ballou

Stresses topics essential to later mathematical work, with theory and problems well balanced. Explanations and illustrations clear and detailed. Classroom-tested; eminently teachable. $2.40

Prices Subject to Discount

GINN AND COMPANY **Boston 17** **New York 11** **Chicago 16** **Atlanta 3** **Dallas 1** **Columbus 16** **San Francisco 5** **Toronto 5**

HART, WILSON, AND TRACEY'S

FIRST YEAR COLLEGE MATHEMATICS

Essentials of College Algebra; Plane and Spherical Trigonometry with Applications; Analytic Geometry; Tables. *886 pages. $4.00*

BRIEF EDITION (*HART*)

The complete edition with Analytic Geometry omitted. *606 pages. $3.00*

For economy, reliability, and ease of reference

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

Outline of the History of Mathematics

by RAYMOND CLARE ARCHIBALD

Fifth edition, June 1941, ii, 76 pages

THIS thoroughly revised and considerably enlarged edition is published by the Mathematical Association to meet a constant demand for up-to-date information on the History of Mathematics, not available in any other single English work. The syllabus, twenty pages of references to other material and sources, and an entirely new Index of Names, provide an excellent basis for a teacher to conduct a semester or year course in this field, or for a student wishing greatly to extend his knowledge. Earlier editions of the Outline have been reviewed very favorably throughout the world. Our expectation is that this fifth edition published within ten years, and more fundamentally revised and extended than any other edition, will continue to meet a need in this country and elsewhere.

Price 75 cents a copy, postpaid, remittance with order

No discount in price to anyone

MATHEMATICAL ASSOCIATION OF AMERICA
McGRAW HALL, CORNELL UNIVERSITY
ITHACA, NEW YORK

THE CARUS MONOGRAPHS

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935; Fourth Impression, 1944.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930; Third Impression, 1943.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936; Fourth Impression, 1943.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSORS DAVID EUGENE SMITH and JEKUTHIEL GINSBURG. (First Impression, 1934.)

No. 6. *Fourier Series and Orthogonal Polynomials,* by PROFESSOR DUNHAM JACKSON. (First Impression, 1941.)

No. 7. *Vectors and Matrices,* by PROFESSOR C. C. MACDUFFEE. (First Impression 1943.)

Price $1.25 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary, McGRAW HALL, Cornell University, ITHACA, N.Y.

Additional copies for members, and copies for non-members, may be purchased from the Open Court Publishing Co., La Salle, Illinois, at the regular price of $2.00 per copy.

LATIN AUTHORS

Westinghouse Electric & Manufacturing Co.

WANTED—Man familiar with statistical methods for controlling quality of products during manufacturing. Must be able to collect and analyse data and make suggestions to improve quality of product and reduce rejects. Must be willing and able to work with shop people. Familiarity with machine tools, gages and mechanical operations desirable. Should have degree in mathematics or physics or equivalent. This is a permanent position with excellent opportunities for advancement during and after the war period. Write the Supervisor, Technical Employment, Union Bank Building, Pittsburgh 22, Pennsylvania.

Westinghouse Electric & Manufacturing Co.

Are you familiar with

Raymond W. Brink's

A FIRST YEAR OF COLLEGE MATHEMATICS

Price, $3.75

A FIRST YEAR OF COLLEGE MATHEMATICS WITH SPHERICAL TRIGONOMETRY

Price, $4.00

D. APPLETON-CENTURY COMPANY

35 West 32nd Street **New York 1, New York**

McGraw-Hill Books of Timely Interest

METHODS OF ADVANCED CALCULUS

By PHILIP FRANKLIN, Massachusetts Institute of Technology. 486 pages, $4.50

Covers those aspects of advanced calculus that are most needed in applied mathematics, including Taylor's series, partial differentiation, applications to space geometry, integration, special higher functions, Fourier series, vector analysis, and the calculus of variations.

MODERN OPERATIONAL MATHEMATICS IN ENGINEERING

By RUEL V. CHURCHILL, University of Michigan. 305 pages, $3.50

Partial differential equations of engineering and Laplace transforms are the two principal topics treated. Problems in ordinary differential equations and other types of problems are also included. The operational properties of the Laplace transformation are derived and carefully illustrated and are used to solve problems in vibrations and resonance in mechanical systems, conduction of heat, electricity, potential, etc.

MATHEMATICAL AND PHYSICAL PRINCIPLES OF ENGINEERING ANALYSIS

By WALTER C. JOHNSON, Princeton University. 343 pages, $3.00

Presents the essential physical and mathematical principles and methods of attack that underlie the analysis of many practical engineering problems. The book emphasizes fundamentals and physical reasoning, and devotes considerable attention to the use of assumptions, procedures in setting up equations, accurate and quantitative reasoning, etc.

MATHEMATICS. *New second edition*

By JOHN W. BRENEMAN, The Pennsylvania State College. *The Pennsylvania State College Industrial Series.* 225 pages, $1.75

Offers brief treatments of simple operations and fractions, ratio and proportion, areas and volumes of simple figures, tables, formulas, fundamentals of algebra, geometrical constructions, trigonometry, etc. Revisions in this edition consist largely of restatement of principles.

HANDBOOK OF AIR NAVIGATION

By W. J. VANDERKLOOT, Captain, R.A.F. Transport Command. 330 pages, $3.50

The first complete manual on air navigation based on the standards required by the International Committee of Air Navigation for the carriage of passengers between foreign countries. All theoretical requirements are covered in a nontechnical manner, together with practical use of the various factors. A feature of the book is the treatment of the value of two position line and three position line fixes.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, INC.

330 West 42nd Street **New York 18, N.Y.**

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

ANTIVIOLENCE

ECHNOLOGY DEPT.

LIBRARY
NOV 1 1944
DETROIT

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 51 NUMBER 8

CONTENTS

Biographies and Collected Works of Mathematicians T. J. Higgins 433
Tangent Triangles to a Biquadratic Curve . . . . J. S. Frame 445
Three-Line Latin Rectangles . . . . . John Riordan 450
Mathematics for Students of the Humanities . . Oystein Ore 453
Clubs and Allied Activities. 458
Discussions . Tomlinson Fort, M. F. Smiley, Michael Wilensky 462
Recent Publications. 469
Problems and Solutions. 472
News and Notices 480
War Information. 483
The Asbury Park Pre-V-12 Program 483
Recircularization of Registrants in the National Roster . . . 486
The Section on Education in the G. I. Bill 486
The Mathematical Association of America 489
Nebraska Section Meeting 489
Kentucky Section Meeting 491
Calendar of Future Meetings. 492

OCTOBER 1944